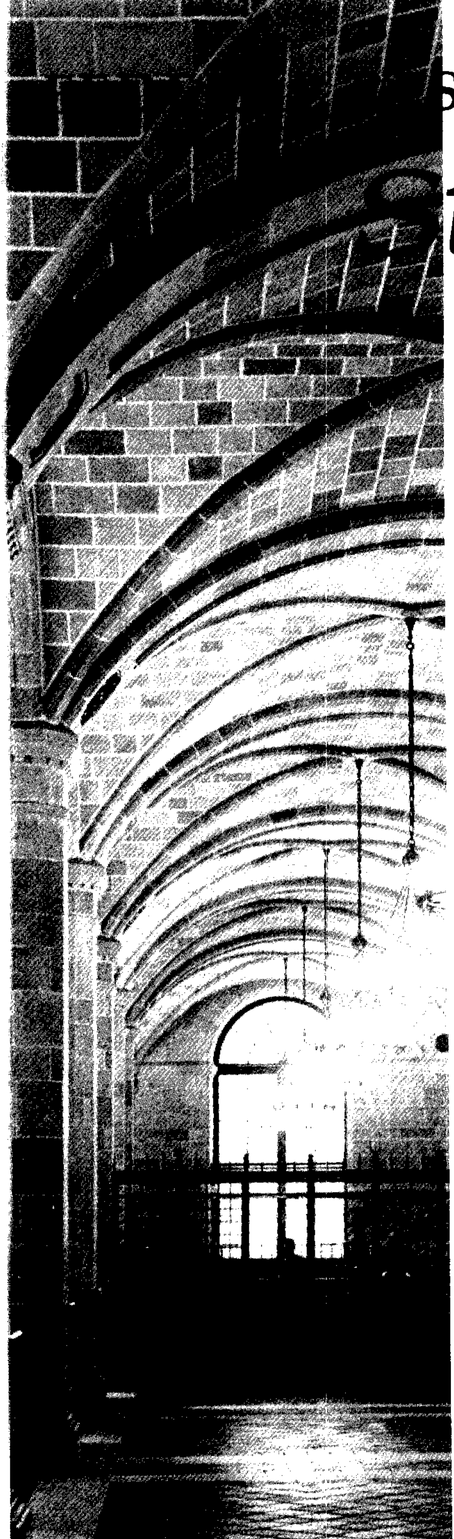

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Option-implied Probability Distributions and Currency Excess Returns

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Abstract

This paper describes a method of extracting the risk-neutral probability distribution of future exchange rates from option prices. In foreign exchange markets, interbank option pricing conventions make possible reliable inferences about risk-neutral probability distributions with relatively little data. Moments drawn from risk-neutral exchange rate distribution are used to explore several issues related to the puzzle of excess returns in currency markets. Tests of the international capital asset pricing model using risk-neutral moments as explanatory variables indicate that option-based moments have considerably greater explanatory power for excess returns in currency markets than has been found in earlier work. Tests of several hypotheses generated by the peso problem approach indicate that jump risk measured by the risk-neutral coefficient of skewness can explain only a small part of the forward bias. These tests take into account not only the second, but the third and fourth moments of the exchange rate implied by option prices, and avoid testing a joint hypothesis including a distributional assumption.

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1 Introduction

Market participants and policy makers require up-to-date information on market sentiment and market beliefs about the future. This information can be compactly expressed in terms of the probability distributions of financial asset prices. The prices of the assets themselves and the prices of derivative contracts on the assets contain much of that information. A classic example is the use of the forward exchange rate as an indicator of the market consensus on the first moment of the future exchange rate; another is the use of the term structure of interest rates as an indicator of the expected future inflation rate.

The payoffs on derivatives such as options are conditional on the future prices of the underlying assets, and therefore reflect market beliefs about the future prices of the underlying assets. This paper describes a method for extracting the risk-neutral probability distribution of future exchange rates from option prices which exploits the identity of the second derivative of a call option's value with respect to the exercise price and the risk-neutral probability density function. We use a relatively small amount of data—essentially, three different exercise prices—to draw inferences about the complete probability distribution. In foreign exchange markets, most option trading takes place over-the-counter, and we will see, the way these markets package options and the pricing conventions they employ make it possible to draw reliable inferences about risk-neutral probability distributions using these few data points.

Risk-neutral probability distributions may shed some light on a long standing puzzle in exchange rate economics, the forward rate bias. This paper reports tests of the international capital asset pricing model (CAPM) using risk-neutral moments as explanatory variables indicate that option-based moments have considerably greater explanatory power for excess returns in currency markets than has been found in earlier work. We also find that investors can earn excess returns from holding currencies for which option prices indicate positive skewness, although such a strategy may be quite risky. These tests take into account the higher moments of the exchange rate implied by option prices, and avoid testing a joint hypothesis including a distributional assumption.

2 Option prices and probability distributions

The payoff at maturity to a European call option with an exercise price X, maturing at time T, is $\max(S_T - X, 0)$, where S_T represents the terminal, or time-T, asset price. Denoting by $c(t, X, T)$ the observed market value at time t of a European call option, we have

$$\begin{aligned} c(t, X, T) &= e^{-r\tau} E^*[\max(S_T - X, 0)] \\ &= e^{-r\tau} \int_X^\infty (S_T - X)\pi(S_T) dS_T, \end{aligned} \tag{1}$$

where S_t represents the time-t asset price, $\tau \equiv T - t$ the time to maturity, r the domestic risk-free continuously compounded discount rate (assumed constant for expositional purposes only), E^* a conditional expectation taken under the risk-neutral probability measure, and $\pi(x)$ the risk-neutral probability density function of the terminal asset price S_T conditional on S_t.

In this context, we are not using equation (1) to discover option prices. Rather, we use equation (1) together with observed market option prices $c(t, X, T)$ to draw inferences on $\pi(x)$. Figure 1 illustrates the relationship between the risk-neutral density and call prices.

The risk-neutral probability density function is the second derivative of the market call price with respect to the exercise price.[1] The first derivative of $c(t, X, T)$ is

$$\frac{\partial c(t, X, T)}{\partial X} = -e^{-r\tau}[1 - \Pi(X)], \tag{2}$$

where $\Pi(S_T)dS_T = P^*\{S_T \leq x\}$ is the risk-neutral cumulative distribution function and $\pi(S_T)$ is the risk-neutral probability density function. The second derivative is

$$\frac{\partial^2 c(t, X, T)}{\partial X^2} = e^{-r\tau}\pi(X). \tag{3}$$

This result reflects the fact that combinations of options can be used to construct claims on an asset which pay off if the realized future asset price falls in a narrow range. Such a combination of

[1]This fact was first noted by Breeden and Litzenberger (1978).

options must be priced as if it were a lottery ticket paying off if the particular range for the future asset prices is realized. The prices of the combinations of options then have the mathematical structure of a probability measure on the future asset price (Dreze 1970, p. 149ff.).

Implementing this approach is difficult because it requires, in principle, options on a continuous (or at least closely-spaced) series of exercise prices on the asset price axis. In practice, not enough option contracts with different exercise prices on a given asset with a given maturity trade simultaneously. Previous authors have made strong modeling assumptions, usually about the parametric family to which the risk-neutral distribution belongs, or used various methods to interpolate between the relative handful of observed options prices to increase the number of exercise prices.[2]

3 The Black-Scholes model and its implied probability distribution

The language and conventions of currency option trading are drawn from the Black-Scholes model, even though neither traders nor academics believe in its literal truth.[3] The model assumes that the spot exchange rate follows a geometric Brownian motion process

$$dS_t = (r - r^*)S_t dt + \sigma S_t dZ, \tag{4}$$

where dZ is the increment to a standard Brownian motion, and that the domestic (foreign) risk-free continuously compounded discount rate r (r^*) and the volatility parameter σ are known constants. Logarithmic exchange rate returns are then normally distributed. The model results in formulas for European put and call values. That for a currency call is

[2]Examples include Shimko (1993), Bates (1996a), Bates (1996b), Rubinstein (1994), Neuhaus (1995), Bahra (1996), Melick and Thomas (1996) and Malz (1996). Söderlind and Svensson (1997) and Jondeau and Rockinger (1997) compare different methods.

[3]The application to options on currencies is often referred to as the Garman-Kohlhagen model, following its exposition in Garman and Kohlhagen (1983).

$$v(S_t,\tau,X,\sigma,r,r^*) = e^{-r^*\tau}S_t\Phi\left[\frac{\ln\left(\frac{S_t}{X}\right)+\left(r-r^*+\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right] - e^{-r\tau}X\Phi\left[\frac{\ln\left(\frac{S_t}{X}\right)+\left(r-r^*-\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right], \tag{5}$$

where $\Phi(\cdot)$ represents the standard cumulative normal distribution function.

Market participants use these formulas even though they do not consider the Black-Scholes model a precise description of how exchange rates actually behave. We will follow the market's practice of expressing facts about options and carrying out numerical procedures using formula (5) without accepting the model. To emphasize this detachment of the formula from the model generating it, we refer to it as the Black-Scholes call pricing *function.*

There is a one-to-one relationship between the volatility parameter σ and the Black-Scholes call pricing function for given values of the remaining arguments, so market prices of calls can be expressed either in units of volatility, called the *Black-Scholes implied volatility* in this context, or in currency units. Given a market price $c(t,X,T)$ in currency units, the corresponding implied volatility σ_t can be found by substituting the maturity, exercise price, the observed spot rate, and the τ-period domestic and foreign interest rates r_t^τ and $r_t^{*\tau}$ into (5) and solving

$$c(t,X,T) = v(S_t,\tau,X,\sigma_t,r_t^\tau,r_t^{*\tau}). \tag{6}$$

Dealers quote option prices in terms of implied volatility, calling its units (decimal or percentage points) *vols.* This does not entail endorsement of the model. Prices are determined by the market: the Black-Scholes call pricing function merely transforms them from one metric to another. In the Black-Scholes model, volatility is the risk-neutral standard deviation of logarithmic changes in the forward exchange rate. When dealers use the Black-Scholes function to calculate implied volatility, it becomes a parameter closely related to, but not identical to, the risk-neutral standard deviation. The upper panel of Figure 2 illustrates this convention.

Exercise prices of over-the-counter currency options are often set equal to the forward exchange rate of like maturity, in which case the option is called *at-the-money forward*. Options for which the exercise price equals the spot exchange rate are called *at-the-money outright.*[4] For example, in response to an inquiry about dollar-mark calls, the dealer might reply that "one-month at-the-money forward calls are 12 at 12.5," meaning that the dealer buys the calls at an implied volatility of 12 vols and sells them at 12.5 vols. If a deal is struck, settlement takes place in currency units. The two counterparties agree on what the current forward rate is, and thus the exercise price, and translate the agreed price from vols to marks per dollar of notional underlying value by substituting the current spot rate, one-month domestic and foreign interest rates, and the contractually-agreed maturity, exercise price and vol price into the Black-Scholes function (5).

The rate of change of the Black-Scholes call pricing function with respect to the spot exchange rate, a fraction called *delta*, is the optimal hedge of an option position under ideal conditions and is referred to by option dealers in managing option price risk:[5]

$$\delta_v(S_t,\tau,X,\sigma,r,r^*) \equiv \frac{\partial v(S_t,\tau,X,\sigma,r,r^*)}{\partial S_t} = e^{-r^*\tau}\Phi\left[\frac{\ln\left(\frac{S_t}{X}\right)+\left(r-r^*+\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right]. \tag{7}$$

The interbank currency option market uses the call delta rather than exercise price as a metric for the moneyness of an option. Fluctuations in spot and forward exchange rates may considerably change the value of an option with a given exercise price without changing implied volatilities. Delta as a metric for moneyness abstracts from these variations, so dealers can avoid recalculating their quotes in response to changes in the cash market which are largely irrelevant to them.[6]

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[4]On June 21, 1995, the dollar-mark spot rate was DM 1.3794, the at-the-money forward volatility was 14.3 percent, and one-month forward dollar-mark traded at a discount of 16 points (DM 0.0016). A one-month at-the-money forward call option on one dollar cost DM 0.0227, or 1.6 percent of the underlying value.

[5]Continuing the example in the prior footnote, the delta of an at-the-money forward dollar call against the mark was 50.6 percent on June 21, 1995. A dollar call with an exercise price of DM 1.3787, slightly higher than the forward exchange rate DM 1.3778, would have a delta of exactly 50 percent.

[6]They may, of course, need to adjust hedges on existing inventories of options in response to fluctuations in spot and forward rates even if implied volatilities are unchanged.

Often, exercise prices are set so delta equals a round number like 25 or 75 percent. If a customer buys, say, a 25-delta dollar-mark call, the exercise price is calculated by setting the left-hand side of equation (7) equal to 0.25 and solving for X, as illustrated in the lower panel of Figure 2.

The delta of a put is

$$\delta_w(S_t, \tau, X, \sigma, r, r^*) \equiv \frac{\partial w(S_t, \tau, X, \sigma, r, r^*)}{\partial S_t} = 1 - \delta_v(S_t, \tau, X, \sigma, r, r^*). \tag{8}$$

Put-call parity implies that puts and calls with the same exercise price have identical implied volatilities, so the volatility of an x-delta put equals that of an $(1 - x)$-delta call.

4 The volatility smile

4.1 The smile in over-the-counter currency option markets

The Black-Scholes model assumes that all options on the same currency have an identical volatility parameter, regardless of time to maturity and moneyness. However, there are systematic "biases" in implied volatilities of options on most assets:

- Options with the same exercise price but different maturities often have different implied volatilities, giving rise to a term structure of implied volatility. A rising term structure indicates that market participants expect short-term implied volatility to rise or a willingness to pay more to protect against longer-term exchange volatility (Campa and Chang 1995).

- Out-of-the money options on currencies with flexible exchange rates often have higher implied volatilities than at-the-money options, indicating that the market perceives exchange rates to be leptokurtotic or is assuming a defensive posture on large exchange rate moves. Such patterns may be generated if implied volatility is stochastic or if the exchange rate follows a jump-diffusion (Heynen 1994, Taylor and Xu 1994).

- Out-of-the-money call options often have implied volatilities which differ from those of equally out-of-the-money puts, indicating that the market perceives the distribution of future exchange rates to be skewed or a willingness to pay more for protection against sharp currency moves in one direction than in the other. Such patterns are generated if the exchange rate follows an asymmetric jump-diffusion (Bates 1991, Bates 1996a).

The latter two phenomena are known as the *volatility smile.* Figure 3 displays a volatility smile exhibiting both skewness and kurtosis, constructed from over-the-counter currency option data by the method to be described below in subsection 4.2. In this example, out-of-the-money options have a higher volatility than at-the-money options, and out-of-the-money call options have higher volatilities than in-the-money calls.[7] The top panel graphs the smile as a function of the put delta and the bottom panel as a function of exercise price.

Empirical studies of the stochastic properties of exchange rates (Boothe and Glassman 1987, Hsieh 1988, de Vries 1994) indicate that the hypothesis that the exchange rate follows geometric Brownian motion is only approximately true. The distributions of nominal returns appear to be leptokurtotic, skewed, and time-varying. There is strong evidence that flexible exchange rate returns follow jump-diffusions, that is, a sum of geometric Brownian and Poisson-distributed jump components, which can account both for the kurtosis and the skewness in nominal returns (Akgiray and Booth 1988, Jorion 1988, Tucker and Pond 1988). If jumps in either direction are equally likely, then the frequency of large changes will be greater than is consistent with normality, but no skewness will be apparent. If jumps in one direction are larger or more frequent, the distribution will be skewed. A time-varying variance of exchange rate returns can be represented by autoregressive conditional heteroscedasticity models, which can account for kurtosis as well as for the time variation of volatility (Hsieh 1988, Hsieh 1989, Baillie and Bollerslev 1989).

The market's awareness of these non-normalities and the relationship of non-normalities to risk appetites and agents' portfolios are reflected in the volatility smile and in the term structure of

[7]Because of relationship (8) between put and call deltas, this is equivalent to saying that out-of-the-money call options have higher volatilities than out-of-the-money puts. The upper panel has put deltas on the x-axis in order to line up better with the bottom panel: a high put delta corresponds to a high exercise price in currency units. Presented this way, the graph is a mirror image of its appearance with call deltas on the x-axis. See Hull (1993, pp. 436ff.) for graphs illustrating the relationship between probability distributions and volatility smiles.

implied volatility. The market may price large exchange rate moves or large moves in a particular direction as likelier to occur than if exchange rates were distributed lognormally. The market nonetheless contentedly uses the Black-Scholes terminology, raising or lowering the implied volatility of options with particular exercise prices to reflect its probability beliefs as well as its appetite for risk. Implied volatility would be a constant across exercise prices in the Black-Scholes environment, so it is a convenient way of pricing departures from the Black-Scholes environment.

We can summarize the observed volatility smile at time t for call options with maturity τ in the function $\sigma_X(t, X, T)$, which gives the implied volatility corresponding to each exercise price for which an option price can be observed. It is related to observed option prices and to the Black-Scholes call pricing formula by

$$c(t, X, T) = v(S_t, \tau, X, \sigma_X(t, X, T), r_t^\tau, r_t^{*\tau})). \tag{9}$$

Few options with different exercise prices trade actively at one time, so $\sigma_X(t, X, T)$, which like $c(t, X, T)$ can be thought of as a table of actual observations, is not continuous.

Option dealers are essentially in the business of trading features of the risk-neutral probability distribution and have developed instruments—option combinations—and market institutions to facilitate this. The most common in the interbank currency option markets is the *straddle*, a combination of an at-the-money forward call and an at-the-money forward put with the same maturity. There is also active trading in option combinations with which participants can take positions on the higher moments of the risk-neutral distribution, particularly the *strangle* and the *risk reversal*, both consisting of an out-of-the-money call and out-of-the-money put. In the latter two combinations, the exercise price of the call component is higher than the current forward exchange rate, and the exercise price of the put is lower. Risk reversals and strangles are usually standardized as combinations of a 25-delta call and a 25-delta put. Figure 4 displays the payoff profiles at maturity of these instruments.

In a strangle, the dealer sells or buys both out-of-the-money options from the counterparty. Dealers usually quote strangle prices by stating the average implied volatility at which they buy or sell the options and record strangle prices as the spread of the strangle volatility over the at-the-money forward volatility. If market participants were convinced that exchange rates move lognormally, the out-of-the-money options would have the same implied volatility as at-the-money options and strangle prices would be zero. Strangles, then, indicate the degree of curvature of the volatility smile and of kurtosis priced into the market.

In a risk reversal, the dealer exchanges one of the options for the other with the counterparty. Because the put and the call are generally not of equal value, the dealer pays or receives a premium for exchanging the options. expressed as the difference between the implied volatilities of the put and the call. The dealer quotes the implied volatility differential at which he is prepared to exchange a 25-delta call for a 25-delta put. For example, if dollar-mark is strongly expected to fall (dollar depreciation), an options dealer might quote dollar-mark risk reversals as follows: "one-month 25-delta risk reversals are 0.8 at 1.2 mark calls over." This means he stands ready to pay a net premium of 0.8 vols to buy a 25-delta mark call and sell a 25-delta mark put against the dollar and charges a net premium of 1.2 vols to sell a 25-delta mark call and buy a 25-delta mark put. If the market perceived exchange rates to be lognormally distributed, the prices of risk reversals would be zero. If the market is postured defensively against significant exchange rate moves in one direction, risk reversals prices are nonzero.[8]

The midpoint of the time-t one-month strangle price can be expressed as

$$str_t = 0.5(\sigma_t^{(0.75)} + \sigma_t^{(0.25)}) - atm_t, \qquad (10)$$

and the risk reversal price as

$$rr_t = \sigma_t^{(0.25)} - \sigma_t^{(0.75)}, \qquad (11)$$

[8]Risk reversal prices of zero do not indicate that no skew is perceived in exchange rates, since a modest skew is implied by lognormality.

where str_t, rr_t, and atm_t denote the one-month strangle price, risk reversal price, and at-the-money volatility, in vols, and $\sigma_t^{(0.25)}$ and $\sigma_t^{(0.75)}$ refer to the implied volatilities of the one-month 25-delta call and 25-delta put. Note that str_t and rr_t are quoted as vol spreads, while atm_t is a vol level.

Using these definitions, the market quotes for the strangle price, risk reversal price, and at-the-money volatility can be solved for observed market prices $\sigma_t^{(0.25)}$ and $\sigma_t^{(0.75)}$:[9]

$$\begin{aligned} \sigma_t^{(0.25)} &= atm_t + str_t + 0.5rr_t \\ \sigma_t^{(0.75)} &= atm_t + str_t - 0.5rr_t \end{aligned} \tag{12}$$

The market one-month implied volatility schedule $\sigma_X(t, X, T)$ can be expressed in terms of delta as $\sigma_t^{(\delta)}$, which contains the same information. The exercise price $X_t^{(0.25)}$, for example, of a one-month 25-delta call option can be calculated by setting $\delta_v(S_t, \tau, X_t^{(0.25)}, \sigma_t^{(0.25)}, r_t^{\tau}, r_t^{*\tau}) = 0.25$. Only a change of units is involved, so $\sigma_t^{(0.25)} \equiv \sigma_X(t, X_t^{(0.25)}, T)$. For any delta

$$\sigma_t^{(x)} \equiv \sigma_X(t, X_t^{(x)}, T), \qquad x \in [0, 1], \tag{13}$$

where $X_t^{(x)}$ satisfies

$$\delta_v(S_t, \tau, X_t^{(x)}, \sigma_t^{(x)}, r_t^{\tau}, r_t^{*\tau}) = x, \qquad x \in [0, 1]. \tag{14}$$

Note also that the implied volatilities of a call with a delta of x and a put with a delta of $x - 1$ are identical, so we can readily include the observed implied volatilities of puts in $\sigma_t^{(\delta)}$. We translate the quote on a one-month x-delta call option from vols into currency units via

$$c(t, X_t^{(x)}, T) \equiv v(S_t, \tau, X_t^{(x)}, \sigma_t^{(x)}, r_t^{\tau}, r_t^{*\tau}), \tag{15}$$

where $X_t^{(x)}$ is calculated from equation (14).

[9]Continuing the example in footnotes 4 and 5, the risk reversal price on June 21, 1995 was -1.0 vols and the strangle price was 0.3 vols. The implied volatility of the 25-delta put, 15.1 vols, was significantly different from that of 25-delta calls (14.1 vols). The average volatility of the out-of-the-money options was 14.6 vols, higher than the at-the-money volatility (14.3 vols).

4.2 Estimating the smile with over-the-counter option data

A continuous volatility smile can be constructed by interpolating a particular functional form through the observed market prices of options . We assume the specification

$$\hat{\sigma}_\delta(\delta) = b_0 atm_t + b_1 rr_t(\delta - 0.50) + b_2 str_t(\delta - 0.50)^2, \qquad (16)$$

In this specification, the volatility smile has three components, illustrated in Figure 5: a linear function of the at-the-money volatility, a linear function of the risk reversal price and the deviation of delta from 0.5, and a quadratic function of the strangle price and the deviation of delta from 0.5.

This functional form is the simplest one which captures the basic information about the smile which the three option prices express. The at-the-money or straddle volatility gives the general level of implied volatility; it is a "measure of location" for the volatility smile. The risk reversal price indicates the skew in the volatility smile. The strangle price indicates the degree of curvature of the volatility smile, that is the degree to which the volatilities of out-of-the-money options exceed the at-the-money volatility. Since the shape of any unimodal probability distribution can be characterized in terms of its second, third, and fourth central moments, it is possible to make accurate inferences about the risk-neutral probability distribution with these over-the-counter options.

Solving for the specific values $(b_0, b_1, b_2) = (1, -2, 16)$, as detailed in Appendix A, we have

$$\hat{\sigma}_\delta(\delta) = atm_t - 2rr_t(\delta - 0.50) + 16str_t(\delta - 0.50)^2 \qquad (17)$$

Figure 6 displays typical volatility smiles in markets with various strangle and risk reversal prices.

This functional form has continuous first and second derivatives and minimizes the number of inflection points in the volatility smile. It can be seen as a Taylor approximation to $\sigma_t^{(\delta)}$ at the point $\delta = 0.50$, since the first derivative of the smile is $2rr_t$ and the second derivative is $32str_t$, or as a spline approximation to $\sigma_t^{(\delta)}$ with parabolic endpoints.

4.3 Estimating the probability distribution from the smile

We want to estimate the risk-neutral probability distribution by second-differencing the current prices of call options with respect to their exercise prices, in accordance with equation (3). So far, we have approximated the market volatility smile schedule $\sigma_t^{(\delta)}$. The next step is to get from delta-volatility space to exercise price-volatility space by translating the approximation $\hat{\sigma}_\delta(\delta)$, which relates implied volatility to delta, into an approximated version of $\sigma_X(t, X, T)$, which relates implied volatility to exercise price. The final step will be to get from exercise price-volatility space to an approximation to $c(t, X, T)$. In this procedure, the Black-Scholes function and its derivatives merely translate pricing conventions from the delta-volatility space in which dealers operate to the exercise price-call price space needed to carry out our calculations.

Delta itself is a function of the implied volatility, so to approximate $\sigma_X(t, X, T)$, we solve equations (17) and (7) simultaneously for implied volatility as a function of exercise price. This leads to the *market implied volatility function*, the implicit function defined by solving

$$\begin{aligned}\sigma = atm_t - 2rr_t &\left\{ e^{-r_t^{*\tau}\tau} \Phi \left[\frac{\ln\left(\frac{S_t}{X}\right) + \left(r_t^\tau - r_t^{*\tau} + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} \right] - 0.50 \right\} \\ + 16str_t &\left\{ e^{-r_t^{*\tau}\tau} \Phi \left[\frac{\ln\left(\frac{S_t}{X}\right) + \left(r_t^\tau - r_t^{*\tau} + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} \right] - 0.50 \right\}^2\end{aligned} \tag{18}$$

for σ. Figure 3 displays an example of $\hat{\sigma}_\delta(\delta)$ and $\hat{\sigma}_X(X)$. The appearance of the market implied volatility function may strike some readers as unusual because it has segments which are concave to the origin and has a slope of zero for large and small X. Appendices C and D show that these features are consistent with the no-arbitrage conditions on call option prices and with empirical work suggesting that observed market volatility functions are quadratic.

We can now move from exercise price-volatility space to an expression for $c(t, X, T)$ to which we can apply equation (3). Substituting the market implied volatility function into the Black-Scholes function, equation (5), yields the generalized Black-Scholes function

$$\hat{v}(X) = e^{-r_t^{*\tau}\tau} S_t \Phi\left[\frac{\ln\left(\frac{S_t}{X}\right) + \left(r_t^\tau - r_t^{*\tau} + \frac{\hat{\sigma}_X(X)^2}{2}\right)\tau}{\hat{\sigma}_X(X)\sqrt{\tau}}\right] - e^{-r_t^\tau \tau} X \Phi\left[\frac{\ln\left(\frac{S_t}{X}\right) + \left(r_t^\tau - r_t^{*\tau} - \frac{\hat{\sigma}_X(X)^2}{2}\right)\tau}{\hat{\sigma}_X(X)\sqrt{\tau}}\right]. \quad (19)$$

Twice differentiating equation (19) with respect to X and multiplying by $e^{-r_t^\tau \tau}$ yields

$$\hat{\pi}(X) = \frac{\partial^2 \hat{v}(X)}{\partial X^2}, \quad (20)$$

the risk-neutral probability density function. The second derivative of $\hat{v}(X)$ is a bona fide probability density function, that is, it is everywhere nonnegative, and integrates to zero and one for very small and very large X. For a fuller discussion of this issue, see Appendix C.

Figure 7 displays the distribution and density functions for S_T for a particular example in which the spot and forward exchange rates are DM 1.50, the at-the-money volatility is 10 vols, the risk reversal price is -1.5 vols (that is, 25-delta dollar puts have a midpoint implied volatility 1.5 vols higher than 25-delta dollar calls), and the strangle price is 0.5 vols (that is, the 25-delta options have on average a midpoint implied volatility 0.5 vols higher than at-the-money options). The example is presented in three panels, showing the effect of the skew expressed in the risk reversal price and the kurtosis expressed in the strangle price separately (panels 1 and 2) and in combination (panel 3). Each panel compares the example with the "benchmark" lognormal case.[10] Figure 8 conveys intuition about how the straddle, risk reversal and strangle prices relate to the implied probability density function. Appendix B describes a way of simplifying the procedure by casting it in terms of the exercise price-forward rate ratio.

Equation (20) can also be used to calculate the moments of the risk-neutral distribution. The risk-neutral first moment is $F_{t,T}$. The r-th central moments is defined by

[10] This may be somewhat misleading, since a nonzero risk reversal price induces some kurtosis, even with a zero strangle price, and a nonzero strangle price increases or offsets the lognormal skew, even with a zero risk reversal price.

$$\hat{\mu}_t^{(r)} = \int_0^\infty (X - F_{t,T})^r \hat{\pi}(X) dX. \tag{21}$$

The risk-neutral annualized standard deviation of percent changes in the exchange rate

$$\hat{\sigma}_t \equiv \sqrt{\frac{\hat{\mu}_t^{(2)}}{\tau}}$$

is general fairly close to atm_t, as can be seen from the summary statistics in Table 2.[11] Of greater interest are the risk-neutral coefficient of skewness

$$\hat{\alpha}_t \equiv \frac{\hat{\mu}_t^{(3)}}{\left[\hat{\mu}_t^{(2)}\right]^{\frac{3}{2}}}$$

and coefficient of kurtosis

$$\hat{\kappa}_t \equiv \frac{\hat{\mu}_t^{(4)}}{\left[\hat{\mu}_t^{(2)}\right]^2} - 3.$$

To illustrate, Figures 9 through 11 display daily spot exchange rates and coefficients of skewness and kurtosis, calculated from the one-month risk-neutral probability distribution, for dollar-mark, sterling-mark, and dollar-yen from 1992 to 1996. Some noteworthy episodes in recent exchange market history are reflected.

- The European Monetary System (EMS) crisis of 1992 and 1993 were accompanied by high positive skewness (dollar bullishness) and kurtosis in the dollar-mark relationship, but had less impact on dollar-yen.

- In September 1992, just before sterling was unpegged from the mark, sterling-mark displays highly negative skewness (sterling bearishness) and extremely high kurtosis. Option protection against a sharp *appreciation* of sterling against the mark grew very costly in early September 1992, although still not nearly as costly as depreciation protection.

[11]The presence of significant kurtosis and skewness modestly increases the standard deviation.

- The dollar swung sharply against other major currencies in 1995. Early in the year, the dollar's decline to record lows against the mark and yen was highly negative skewness (dollar bearishness) and high kurtosis. Later in the year, the dollar's recovery is accompanied by sharply positive skewness and unusually low—occasionally even negative—kurtosis.

I checked the accuracy of the procedure in two ways. One was to compare the option prices generated by the interpolation for deltas other than 25, 50, and 75 with market price data. As elucidated in Appendix D, the implied volatilities quotes of some dealers for deep in- and out-of-the-money options do occasionally differ from one another and from the interpolation. However, these differences lead to negligible differences in call values, and thus the risk-neutral distribution. The second check was to compare the parameterization of the volatility smile proposed here with an alternative which includes an additional quartic term in the strangle price which can potentially produce a distribution with higher skewness and kurtosis. However, for a wide range of option prices, the risk-neutral distributions generated by the two specifications did not significantly differ.

5 Data

Currency option price data was provided by over-the-counter currency option dealers and include daily indicative levels for one-month at-the-money option volatilities (straddle prices) and one-month 25-delta risk reversal and strangle prices from March 31, 1992 through June 11, 1996, for sterling-dollar (cable), dollar-mark, dollar-yen, mark-yen, sterling-mark, and sterling-yen. The exchange rates and one-month Eurodeposit rates are from the BIS Data Bank.[12]

The currency of denomination is the domestic currency and the currency on which the option is written is the foreign currency, in conformity with market convention. For example, a dollar call against the mark is a call on one dollar, with an exercise price denominated in marks, and a sterling call against the mark is a call on one pound, with an exercise price denominated in marks. Table 1

[12]Eurodeposit rates are converted to continuously compounded annual rates by taking the logarithm of one plus the Eurodeposit rate. Forward exchange rates are calculated from the Eurodeposit rates, which are collected at the same time of day as the exchange rates. The option data is collected about two hours later, at London noon.

summarizes these quotation conventions. Table 2 presents descriptive statistics on the data.

6 Implied probability distributions and the forward bias puzzle

6.1 The forward bias puzzle

The probability distributions implied by option prices may shed light on a long standing puzzle in exchange rate economics, the forward rate bias, which may be characterized in several ways: financial instruments which are similar in every respect except that they are denominated in different currencies do not have identical returns when expressed in a common currency; the forward foreign exchange rate is not a conditionally unbiased predictor of the future spot exchange rate; and predictable currency returns are not zero.

To isolate the market expectation on the future spot rate from preference factors, analysis of the forward bias is framed in terms of the relationship

$$E_t[s_T] = f_{t,T} + p_t, \tag{22}$$

where s_t and $f_{t,T}$ equal the logarithms of the spot and forward exchange rates and $p_t \equiv E_t[s_T] - f_{t,T}$ is the predictable logarithmic return on an open currency position (Fama 1984).

Under rational expectations, the market's subjective expectation and the conditional mathematical expectation of the future exchange rate are equal. The deviation of the conditional forecast from the realized exchange rate should average zero over a long period and exhibit no correlation with elements of the conditioning set. Under the additional hypothesis of risk-neutrality, the market's subjective expectation of the future exchange rate is equal to the risk-neutral expectation, the forward exchange rate. This joint hypothesis is generally tested with the Fama regression

$$s_T - s_t = \beta_0 + \beta_1(f_{t,T} - s_t) + \epsilon_T, \tag{23}$$

with ϵ_T a mean-zero variate uncorrelated with variables dated t or earlier. The null hypothesis is that β_0 is equal to a possibly non-zero constant (including a Jensen's inequality term), and $\beta_1 = 1$.

Tests generally lead to a rejection of the joint hypothesis, since estimates of β are far from unity. The frequent finding of significantly negative β values for dollar-based exchange rates indicates, at least for dollar rates, that if expectations are rational, the variance of the risk premium is considerably greater than the variance of the expectation error. Table 3 shows the results of the Fama regression for the six currency pairs studied here during the 1992-1996 period; the forward premium has little if any explanatory power for future exchange rate changes. Negative coefficients are found for the three currency pairs involving the Japanese yen. Japanese money market rates dropped sharply over the sample period, so the forward premium of the yen vis-à-vis other currencies is greatest late in the sample period. The period of the yen's most rapid appreciation was, however, early in the sample period; late in the sample period the yen depreciated rapidly.

These results have led to a search for explanations of the high relative variance of the risk premium, surveyed by Lewis (1995) and Engel (1996). Proposed explanations amend one or more elements of the joint hypothesis underlying the Fama regression (23) by positing (i) a time-varying risk premium susceptible to treatment in a partial or general equilibrium capital asset pricing model (CAPM), (ii) small-sample expectational errors, in particular, peso problems or drawn-out learning by agents, which do not violate the rational expectations hypothesis but bias the ordinary least squares estimate of β, (iii) bona fide departures from rational expectations, (iv) transient imbalances in the supply and demand for currency caused by the market's inability to instantly "digest" large transactions, or (v) serial correlation of excess returns induced by central bank monetary operations.[13] The remainder of this section will explore whether risk-neutral exchange rate moments drawn from option prices can bring evidence to bear on the first two approaches.

6.2 Option prices and the risk premium

A widely used class of models of the exchange rate risk premium is based on a CAPM derived in a static mean-variance or stochastic intertemporal framework, resulting in an expression

[13]See McCallum (1994) on domestic and Arnott and Pham (1993) on foreign monetary operations.

$$p_T = g(\mathbf{x}_t, \Omega_t), \tag{24}$$

for predictable excess returns where Ω_t is the market's perceived conditional variance-covariance matrix of log changes in spot exchange rates, and $\mathbf{x}_t$ is a vector of variables which may include outstanding supplies of assets denominated in different currencies, the shares in consumption of goods invoiced in different currencies, and the covariances of foreign deposit returns with domestic-currency inflation or interest rates.[14] To test the CAPM, one regresses realized excess returns

$$\xi_T \equiv \frac{p_t + \eta_T}{\tau} = \frac{s_T - f_{t,T}}{\tau} = r_t^{*\tau} - r_t^{\tau} + \frac{s_T - s_t}{\tau}, \tag{25}$$

where $\eta_T \equiv s_T - E_t[s_t]$ is the forecast error, on a set of explanatory variables included in $g(\mathbf{x}_t, \Omega_t)$.

In this framework, Lyons (1988) represents elements of Ω_t by squared implied volatilities from prices of currency options traded on the Philadelphia Stock Exchange (PHLX). Because these are American options, which permit early exercise, Lyons uses a binomial pricing model rather than the Black-Scholes function to extract the implied volatilities.[15] Data on the remaining arguments of the model—simultaneous domestic and foreign interest rates and spot exchange rates—are drawn from exchange sources and publicly available U.S. treasury bill quotes. He chooses the options closest to at-the-money outright among contracts with different exercise prices for a given maturity, and options with a remaining time to expiry between three and six months.

Lyons focuses on cable, dollar-mark, and dollar-yen, representing the diagonal elements of Ω_t by the squares of the implied volatilities $\sigma_{£\$,t}$, $\sigma_{\$DM,t}$, and $\sigma_{\$¥,t}$. Because exchange-traded cross-rate options were unavailable, the covariances of the three exchange rates against the dollar were represented by $\rho_{ij,t}\sigma_{i,t}\sigma_{j,t}$, $i \neq j$, where $\rho_{ij,t}$ is the historical correlation of dollar exchange rates i and j over the entire sample period for $i, j =$ £\$,\$DM,\$¥.

[14] See, for example, Frankel (1982), Frankel (1986), and Giovannini and Jorion (1987).

[15] Only the option design differs: the distributional assumption underlying the binomial model, that the exchange rate follows geometric Brownian motion, is identical with that of the Black-Scholes model.

Lyons's finding that elements of Ω_t represented by implied volatilities had significant explanatory value for excess returns tended to support the CAPM approach to the risk premium. His work can be extended in two ways, which we will explore in the remainder of this subsection. First, using over-the-counter option price data makes it possible to avoid several potential sources of data error. Second, third and fourth exchange rate moments (skewness and kurtosis) can be incorporated into the estimates in addition to the second moments.

Using over-the-counter data has several advantages over exchange-traded data:

- Lyons adopts the Black-Scholes model, in which the risk-neutral and subjective second moments are identical, that is, there is no volatility risk premium. He thus tests a joint hypothesis which includes the distributional assumption that the exchange rate follows the stochastic process (4). By calculating the risk-neutral variance from an estimate of the entire risk-neutral probability distribution, we avoid asserting any auxiliary distributional hypothesis.

- Because the exchange-traded options are not exactly at-the-money, variations in the level of implied volatility calculated from them are commingled with variations in the curvature of the volatility smile. At-the-money forward over-the-counter options avoid this distortion.

- Fresh over-the-counter options with some standard maturities are issued daily, so a time series of constant-maturity one-month implied volatilities can be constructed and aligned with data on one-month excess currency returns. This avoids the maturity mismatch between the implied volatilities Lyons calculates and his excess return data and the distortion induced by commingling variations in the term structure of implied volatility and in the level of volatility.

- Data for cross-rate options make it possible to use the exact analog of implied variance for the implied covariances of the dollar exchange rates rather than a sample correlation. The implied covariance can be calculated from the at-the-money dollar and cross-rate options as

$$\text{icov}_{ij,t} = \frac{(atm_{i,t})^2 + (atm_{j,t})^2 - (atm_{ij,t})^2}{2}, \qquad i \neq j, \qquad i,j = \text{£\$,\$DM,\$¥}.$$

Table 4 presents the results of our replication of Lyons' work.[16] Where Lyons found $\bar{R}^2$s between 0.06 and 0.09, we find $\bar{R}^2$s of 0.13 to 0.15. The coefficient of the currency pair's own implied variance is positive for dollar-mark and dollar-yen, and negative for cable. In the CAPM context, this can be interpreted to mean that an increase in uncertainty about the future level of the exchange rate ordains higher equilibrium returns to holders of dollars, that is, an immediate dollar sell-off. This result identifies the dollar as the "weak" currency during the sample period, an observation which is at least plausible for dollar-mark and dollar-yen but less so for cable.[17]

Our second extension of Lyons (1988) is to incorporate information on the third and fourth exchange rate moments by regressing ξ_T on the option-implied risk-neutral standard deviation and coefficients of skewness and kurtosis for each currency pair. The equation we estimate, for each of the three currency pairs dollar-mark, dollar-yen, and sterling-mark, is

$$\xi_{j,T} = \beta_0 + \sum_i \beta_{\hat{\sigma},ij}\hat{\sigma}_{i,t} + \sum_i \beta_{\hat{\alpha},ij}\hat{\alpha}_{i,t} + \sum_i \beta_{\hat{\kappa},ij}\hat{\kappa}_{i,t} + \eta_{j,T}, \quad i,j = (\$£,\$DM,\$¥,DM¥,£DM,£¥) \tag{26}$$

We also estimate a version of equation (26) including the forward premium, since information on the risk-neutral first moment of the exchange rate may contribute to explaining excess returns, and may be correlated with higher moments:

$$\xi_{j,T} = \beta_0 + \sum_i \beta_{\hat{\sigma},ij}\hat{\sigma}_{i,t} + \sum_i \beta_{\hat{\alpha},ij}\hat{\alpha}_{i,t} + \sum_i \beta_{\hat{\kappa},ij}\hat{\kappa}_{i,t} + \beta_{f,j}(f_{j,t,T} - s_{j,t}) + \eta_{j,T}, \quad i,j = (\$£,\$DM,\$¥,DM¥,£DM,£¥) \tag{27}$$

The results are recorded in Table 5. For all six currency pairs, the explanatory value of the risk-neutral moments is quite high, with $\bar{R}^2$ for the version excluding the forward premium between

[16] Because we are estimating one-month prediction equations using daily data, the familiar problem arises that successive equation errors follow a moving-average. The t-statistics reported in Table 4 and successive tables are heteroscedasticity and autocorrelation consistent.

[17] However, as documented in Table 4, short dollar positions against the mark and yen earned negative, not positive excess returns during the sample period, abstracting from transaction costs.

26 and 40 percent. With the exception of sterling-mark, the forward premium is not significant and contributes little to $\bar{R}^2$.[18]

The risk-neutral exchange rate moments are highly correlated with one another and with the forward premium, so inferences concerning the influence of individual moments cannot be made accurately and including different subsets of risk-neutral moments in the regression significantly changes some coefficients. To provide some sense of the coefficients of individual risk-neutral moments, a "typical" such equation is reported in Table 6.

The coefficients are difficult to interpret in the absence of an explicit model that takes higher exchange rate moments into account. Applying the intuition of the CAPM, the regression coefficient of a currency pair's own coefficient of kurtosis should have the same sign as that of the own variance. If those coefficients are positive, it indicates that the market requires a risk premium for agents to willingly hold the supply of outside assets denominated in that currency pair.

If agents dislike abrupt changes in currency values, the regression coefficient of a currency pair's own coefficient of skewness should be positive. Suppose, for example, the dollar has positive skewness against the mark, raising $E[s_{\$DM,T}]$ by increasing the market assessment of the area under the right tail of dollar-mark's density function. Market participants may be reluctant to "buy" the tail event at its full expected value, reducing the risk-neutral expectation $f_{\$DM,t,T}$ below $E[s_{\$DM,T}]$ and resulting in a predictable excess return to long dollar-mark positions. This phenomenon might be amplified (dampened) if, in addition, dollar-denominated assets exceed (fall short of) their share in the minimum-variance portfolio and would require a risk premium even if the distribution were not skewed. We will discuss this regression coefficient further in the next subsection.

For most of the six currency pairs, these priors on the regression coefficients of the own risk-neutral moments are validated, although the coefficients are not always statistically significant. Among the anomalies are the sterling crosses, which may be due to the EMS crisis of 1992. Perceived kurtosis as well as skewness was exceptionally high from August to October 1992, but sterling

[18]Pagès (1996) presents similarly high $\bar{R}^2$ for regressions of dollar-mark, dollar-yen and mark-French franc excess returns on at-the-money volatilities, risk reversal prices, and forward premia, but over a rather short 11-month sample period.

investors were punished rather than rewarded for bearing that risk.[19] A handful of observations pairing high skewness and kurtosis with low excess returns is likely responsible for the result.

6.3 Option prices and the peso problem

A second source of the forward bias on which option-based risk-neutral moments may shed light are rational systematic forecast errors (Lewis 1995) arising from a past policy change about which agents learn gradually, or small-sample biases in the coefficient β of equation (23) arising from future policy changes which affect current asset prices but are not realized frequently enough in-sample.

To frame the discussion, suppose agents expect the exchange rate to follow a jump-diffusion with risk-neutral representation[20]

$$dS_t = (r - r^* - \lambda k_t)S_t dt + \sigma S_t dZ + S_t k_t dq_t, \tag{28}$$

where k_t is the time-t jump size and q_t is a Poisson-distributed random variable with parameter λ. One may think of k_t as a time-varying, but nonrandom discrete percent change in the exchange rate occurring with probability $\lambda\tau$ over the interval $(t, T]$.

The risk-neutral expected value of τ-period percent changes in the exchange rate is still equal to the forward premium $(r - r^*)\tau$, but now includes an expected jump component $\lambda k_t \tau$. If too few jumps are observed during a sample period, the risk-neutral mean percent change in the exchange rate will exceed the forward premium on average by $\lambda k_t \tau$ (or somewhat less), biasing β downwards.

The risk-neutral third moment of $s_T - s_t$, which measures the likelihood of large appreciations relative to depreciations of the same magnitude, equals $[\lambda \ln(1 + k_t)^3]\tau^{-\frac{1}{2}}$ when S_t follows the process (28)(Bates 1991, p. 1023). Thus the risk-neutral coefficient of skewness is a good proxy for

[19]One might expect skewness to have been the dominant feature of sterling-mark implied distributions during the ERM crisis. In fact, skewness was quite high, but not more than for other currency pairs during periods of "normal" weakness, while kurtosis reached unprecedented levels. For further details on the behavior of sterling-mark option prices during the ERM crisis see Malz (1996).

[20]Note that equation (28) is valid only for describing percent changes in the domestic-currency price of foreign exchange, not the foreign-currency price of the domestic currency unit.

$\lambda k_t T$ as a metric for perceived jump risk, suggesting several avenues for exploring whether jump risk affects the relationship between forward premia and subsequent exchange rate realizations. To focus on the coefficient of skewness and avoid the problems of collinearity discussed in previous subsection, these tests are carried out with univariate regressions.[21]

If the exchange rate follows (28), variations in the expected jump component $\lambda k_t T$ are reflected one-for-one in variations of the forward premium and the risk-neutral coefficient of skewness should have a positive relationship with the forward premium. This can be tested with the regression

$$f_{t,T} - s_t = \beta_0 + \beta_1 \hat{\alpha}_t + \eta_T \tag{29}$$

for each currency pair. The results are displayed in Table 7, and provide little confirmation of peso problem effects. For four of the currency pairs, the coefficient is positive, but significant only for dollar-mark and sterling-mark. The three currency pairs involving the Japanese yen have either negative or insignificant coefficients. This result is related to the negative coefficient they show in tests of the Fama regression (see Table 3): the forward premium of the yen against the dollar is highest late in the sample, when option prices indicate a persistent skew in favor of a weaker yen.

A second way to test whether the forward bias is due to peso problem risk is to include α_t in the Fama regression. If unbiasedness is due in substantial part to jump risk, the coefficient of the forward premium should be closer to unity than when κ is omitted. As seen, however, in Table 8, even when jump risk is included in the Fama regression, the coefficient of the forward premium remains negative for three currency pairs involving the yen.

Finally, if the risk that the spot rate may jump between now and the maturity of a forward contract raises the forward rate, and if over the entire sample the increase in the forward premium

[21] The findings here are similar to those of Bates (1996a) using daily Chicago Mercantile Exchange data for dollar-mark and dollar-yen futures options from 1984 to 1992. Bates compared a version of the jump diffusion (28) in which the jump size k_t is not only time-varying but random with the standard diffusion (4) and found that the jump diffusion had greater explanatory power for dollar-mark futures prices, although not for dollar-yen futures prices. However, the additional jump diffusion parameters did not, for either currency pair, affect the rejection of unbiasedness.

induced by jump risk exceeds the realized depreciation attributable to jumps, then the coefficient in a regression of excess returns on the risk-neutral coefficient of skewness should be negative:

$$\xi_T = \beta_0 + \beta_1 \hat{\alpha}_t + \eta_T \tag{30}$$

As shown in Table 9, the coefficient in such a regression is positive for five currency pairs, indicating that the forward premium under- rather than overcompensates for jumps in the sample. This finding suggests that excess returns can be earned by holding currencies for which option markets are pricing in jumps, counter to the peso problem approach, under which the forward premium should rise enough in the presence of positive skewness that expected excess returns are eliminated.

The distinction between the risk-neutral and true distributions is important in this context. As noted in the previous subsection, market participants may require a premium to compensate for jump risk, offsetting the negative relationship between the excess returns and skewness. The market's expectation of jumps raises the forward rate, diminishing excess returns, but risk aversion may dampen the rise in the forward rate, raising excess returns. We cannot distinguish these influences because we are working only with the risk-neutral moments. There may be a peso problem effect on excess returns, but one dominated by investor reluctance to earn excess returns in the form of abrupt surges in currency values.

7 Conclusions

I have presented here a simple way of extracting the implied risk-neutral density function of the future exchange rate from prices of a small number of options with different exercise prices. The main assumption employed is that the volatility smile has a particular functional form. The risk-neutral distributions thus recovered do not rely on additional distributional assumptions. The estimated risk-neutral distributions incorporate phenomena such as stochastic volatility, jump processes, stochastic interest rates, and volatility correlated with exchange rates, so far as they are reflected in actual option prices.

This technique can be of value to market observers and participants. For example, value-at-risk (VAR) techniques for the management of market risks often rely on the assumption that asset prices behave lognormally. One drawback of VAR is that it may inadequately account for the large but rare market moves which can potentially endanger a firm's survival. Practitioners are aware of this and have given much thought to the likelihood of adverse "tail events." Risk-neutral probability distributions can serve as a useful complement to a variance-covariance matrix based on historical data since they are market-based and do not assume a normal distribution.

The risk-neutral exchange rate moments appear to have considerable explanatory value for excess returns in a CAPM framework. The results presented here are less ambiguous than in earlier work by Bates and Lyons because the data used here lend themselves better to carrying out the estimation. As in that earlier work, the caveat must expressed that while the CAPM is framed in terms of subjective moments, the present work tests it with risk-neutral moments.

Our results indicate that investors can earn excess returns from holding currencies for which option prices indicate positive skewness—practically, currencies for which risk reversal prices are positive. One explanation for this finding is that investors are less willing to earn excess returns via jumps in currency values, that is, are unwilling to purchase the tail events at their full expected value. Further work might seek to quantify the risks of such as a strategy.

This result has implications for policy issues which have been tested in the CAPM framework. A potential area for further study is the effectiveness of sterilized foreign exchange intervention, where the typical finding that intervention cannot significantly influence exchange rates by altering relative asset supplies might be revisited using the risk premium model presented here.

Appendices

A Derivation of the volatility smile

To solve for specific values for the parameters b_i, $i = 0, 1, 2$, impose the condition that the observed at-the-money volatility, risk reversal price and strangle price lie exactly on $\hat{\sigma}_\delta(\delta)$. Algebraically, this is done by substituting equation (16) into the definitions (10) and (11).

The at-the-money, or 50-delta volatility[22], sets $b_0 = 1$, since

$$atm_t = \hat{\sigma}_\delta(0.50) = b_0\hat{\sigma}_\delta(0.50) + b_1 rr_t \cdot 0 + b_2 str_t \cdot 0.$$

Similarly, the risk reversal price sets $b_1 = -2$, since

$$rr_t = \hat{\sigma}_\delta(0.25) - \hat{\sigma}_\delta(0.75) = -\frac{b_1}{2} rr_t.$$

Finally, the strangle price sets $b_2 = 16$, since

$$str_t = \frac{\hat{\sigma}_\delta(0.25) + \hat{\sigma}_\delta(0.75)}{2} - atm_t = 0.25^2 b_2 str_t.$$

B Simplified Black-Scholes function

A simplified version of the Black-Scholes call pricing function permits us to calculate the implied probability of percent changes in the dollar's value using only the prices, in vols, of dollar calls and the U.S. interest rate. Define $Q \equiv \frac{X}{F_{t,T}}$, where the forward exchange rate $F_{t,T}$, the price at time t of a claim on one unit of foreign currency deliverable at time T, is related to the spot rate by $F_{t,T} = S_t e^{(r-r^*)\tau}$. Substitute Q into equation (5), multiply it by $e^{r\tau}$, and divide it by $F_{t,T}$ to get

$$v(Q,\tau,\sigma) \equiv \frac{e^{r\tau}}{F_{t,T}} v(S_t,\tau,X,\sigma,r,r^*) = \Phi\left[-\frac{\ln(Q) - \frac{\sigma^2}{2}\tau}{\sigma\sqrt{\tau}}\right] - Q\Phi\left[-\frac{\ln(Q) + \frac{\sigma^2}{2}\tau}{\sigma\sqrt{\tau}}\right], \quad \text{(A.1)}$$

which can be interpreted as the time-T value, rather than time-t value, of a call on one unit of foreign exchange currently priced at unity, with a exercise price of Q. The value of an at-the-money forward dollar call at maturity, expressed as a fraction of the current forward rate, can be found by substituting the at-the-money volatility for σ, the appropriate maturity for τ, and unity for Q. The call delta can be calculated as

[22] We have tacitly set $atm_t = \sigma_t^{(0.50)}$. The at-the-money volatility has, to be precise, a delta slightly different from 50 percent, but the difference is insignificant.

$$\delta_v(Q,\tau,\sigma,r^*) \equiv \frac{\partial v(Q,\tau,\sigma,r^*)}{\partial Q} = e^{-r^*\tau}\Phi\left[-\frac{\ln(Q)-\frac{\sigma^2}{2}\tau}{\sigma\sqrt{\tau}}\right]. \tag{A.2}$$

Next, substitute into equation (A.1) the volatility function $\hat{\sigma}_Q(Q)$, the implicit function defined by solving

$$\begin{aligned}\sigma = atm_t &+ 2rr_t\left\{e^{-r_t^*\tau}\Phi\left[-\frac{\ln(Q)-\frac{\sigma^2}{2}\tau}{\sigma\sqrt{\tau}}\right]-0.50\right\} \\ &+ 16str_t\left\{e^{-r_t^*\tau}\Phi\left[-\frac{\ln(Q)-\frac{\sigma^2}{2}\tau}{\sigma\sqrt{\tau}}\right]-0.50\right\}^2\end{aligned} \tag{A.3}$$

for σ, to get the simplified form of the generalized Black-Scholes call function $\hat{v}(Q)$

$$\hat{v}(Q) = \Phi\left[-\frac{\ln(Q)-\frac{\hat{\sigma}_Q(Q)^2\tau}{2}}{\hat{\sigma}_Q(Q)\sqrt{\tau}}\right] - Q\Phi\left[-\frac{\ln(Q)+\frac{\hat{\sigma}_Q(Q)^2\tau}{2}}{\hat{\sigma}_Q(Q)\sqrt{\tau}}\right] \tag{A.4}$$

One plus the first derivative of equation (A.4) with respect to Q gives the risk-neutral probability $\hat{\Pi}(Q)$ that the exchange rate will end $100(Q-1)$ percent or less above the current forward rate. The second derivative of equation (A.4) with respect to Q is the risk-neutral probability density function $\hat{\pi}(Q)$ of percent changes in the exchange rate.

C No-arbitrage conditions on the generalized Black-Scholes function

The conditions guaranteeing that $\hat{\pi}(Q)$ is a bonafide probability density function are equivalent to no-arbitrage conditions on $\hat{v}(Q)$.[23] The derivative of equation (A.4) with respect to Q is

$$\begin{aligned}\frac{\partial\hat{v}(Q)}{\partial Q} &= \frac{\partial v(Q,\tau,\sigma)}{\partial\sigma}\frac{\partial\hat{\sigma}_Q(Q)}{\partial Q} - \frac{\partial v(Q,\tau,\sigma)}{\partial Q} \\ &= \frac{\partial\hat{\sigma}_Q(Q)}{\partial Q}\sqrt{\tau}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left[\frac{\ln(Q)-\frac{\hat{\sigma}_Q(Q)^2}{2}\tau}{\hat{\sigma}_Q(Q)\sqrt{\tau}}\right]^2} - \Phi\left[-\frac{\ln(Q)+\frac{\hat{\sigma}_Q(Q)^2}{2}\tau}{\hat{\sigma}_Q(Q)\sqrt{\tau}}\right].\end{aligned} \tag{A.5}$$

The risk-neutral cumulative distribution function is

[23] These conditions are framed by Hodges (1996) in terms of the volatility function $\hat{\sigma}_Q(Q)$.

$$P^*\{Q_T \leq Q\} \equiv \hat{\Pi}_t(Q) = 1 + \frac{\partial v(Q,\tau,\sigma)}{\partial \sigma}\frac{\partial \hat{\sigma}_Q(Q)}{\partial Q} - \frac{\partial v(Q,\tau,\sigma)}{\partial Q}. \tag{A.6}$$

To show that the second derivative of $\hat{v}(Q)$ is a probability density function, we must show that

(i) $0 \leq \hat{\Pi}_t(Q) \leq 1 \quad \forall Q$,

(ii) $\lim_{Q\to 0} \hat{\Pi}_t(Q) = 0$,

(iii) $\lim_{Q\to\infty} \hat{\Pi}_t(Q) = 1$.

The second term on the right-hand side of equation (A.6) is the distortion of the Black-Scholes first derivative attributable to the curvature and skew of the volatility smile. It is the product of the derivative of the volatility function $\hat{\sigma}_Q(Q)$ and the option vega, the derivative of the Black-Scholes call pricing function (A.1) with respect to σ. The slope of $\hat{v}(Q)$ is equal to the slope of the simplified Black-Scholes function $v(Q,\tau,\sigma)$ plus or minus this smile-induced distortion term. The components of the smile distortion term are illustrated in panel a of Figure A.1.[24] If $\hat{\Pi}_t(Q)$ fails to satisfy the conditions just enumerated, it will be on account of this second term, so let us examine it in some more detail.

As can be seen from the lower panel of Figure 3, the slope of $\hat{\sigma}_Q(Q)$ is zero at three places, for large and small Q (deep in- and out-of-the-money calls), and for a value of Q close to unity, at the point where

$$\delta_v(Q,\tau,\sigma,r^*) = 0.5 + \frac{rr_t}{16str_t}. \tag{A.7}$$

Both $\frac{\partial \hat{\sigma}_Q}{\partial Q}$ and the option vega $\frac{\partial v}{\partial \sigma}$ and thus the entire second term on the right-hand side of equation (A.6), $\frac{\partial v}{\partial \sigma}\frac{\partial \hat{\sigma}_Q}{\partial Q}$, tends toward zero for large or small Q and are finite everywhere. The third term on the right-hand side of equation (A.6) is a normal cumulative distribution function, so it tends to zero (unity) as Q becomes small (large), proving (ii) and (iii).

Any violation of condition (i) will only occur for intermediate values of Q, since the entire term $\frac{\partial v}{\partial \sigma}\frac{\partial \hat{\sigma}_Q}{\partial Q}$ equals zero for large and small Q. Numerical experiments with a wide range of values of atm_t, rr_t, and str_t verify that $\frac{\partial v}{\partial \sigma}\frac{\partial \hat{\sigma}_Q}{\partial Q}$ is quite small in magnitude relative to $\frac{\partial v(Q,\tau,\sigma)}{\partial Q}$. Some of these

[24] A closed-form expression for $\frac{\partial \hat{\sigma}_Q(Q)}{\partial Q}$ and further details on the derivations in this Appendix are available from the author.

are displayed in panels b and c of Figure A.1. The distortion is greatest when the at-the-money volatility is low, or the magnitude of the risk reversal and strangle prices is high.

If the probability density function is skewed to the left (right), this term is greater (less) than or equal to zero for Q below the value satisfying (A.7), and less (greater) than or equal to zero for Q above the value satisfying (A.7). Figure A.2 displays some examples of how the distortion term is related to the risk-neutral cumulative distribution function. Panel a shows a typical case, in which $atm_t = 0.10$, $rr_t = -0.015$, and $str_t = 0.005$. Panel b displays an extreme case which is conceivable for an Exchange Rate Mechanism currency: $atm_t = 0.03$, $rr_t = -0.03$, and $str_t = 0.01$. Panels c and present examples in which atypically extreme skewness and kurtosis are present: $atm_t = 0.10$, $rr_t = \pm 0.03$, and $str_t = 0.01$. Even for these extreme cases, the cumulative distribution function satisfies conditions (i)-(iii) above. Intuitively, the smile distortion term merely redistributes some probability mass from one side of the distribution to the other by steepening the slope of $\hat{v}(Q)$ on one side of $Q = 1$ (the forward rate) and flattening it on the other.

D Accuracy of the procedure

The accuracy of the method presented here depends on the interpolated volatility smile being a good approximation to actual market prices. How close is the interpolated smile represented by equation (17) to the implied volatilities of options with deltas other than 25, 50, and 75 percent? How close are the call option prices generated by the interpolated smile to the observed prices of options with deltas other than 25, 50, and 75 percent? Most important, what impact do the interpolation errors have on estimates of the risk-neutral distribution?

Because the 25, 50, and 75-delta options anchor the interpolated smile, that equation (17) is very close to actual market implied volatilities for deltas between about 15 and 85. For options with deltas outside that range, so-called "wing options," trading is thinner, and dealers often have rather different quotes for, say, 10- and 90-delta options, making it more difficult to establish "the" market level.

There may therefore occasionally be small differences between simultaneous implied volatility quotes among market for options other than 25-, 50-, and 75-delta and considerable differences for 10- and 90-delta options, in turn implying considerable differences between the implied volatility quotes of at least some dealers and the implied volatilities generated by our interpolation technique. We explored these issues by obtaining, for a few dates, a finer "ladder" of one-month dollar-mark call option volatilities with $\delta = \{0.05, 0.10, \cdots, 0.95\}$.[25]

[25]Major dealers can provide indicative levels of implied volatility for any calls of delta at any time in the trading day. However, very few record the implied volatilities of options with deltas other than 25, 50 and 75 percent.

The left panels of Figure 3 compare our interpolation technique with the ladder provided by two dealers during the New York afternoon of March 6, 1997, chosen to represent the range of approaches to pricing wing options: Dealer I set low and Dealer II set somewhat higher implied volatilities than average. For 15- to 85-delta options, the implied volatility quotes are very close to the interpolated values, and, of course, coincides with it for $\delta = \{0.25, 0.50, 0.75\}$.

These differences in implied volatilities for deltas closer to zero or unity, however, imply only small differences in option prices expressed in currency units, since for very high and low exercise prices, the option vega $\frac{\partial v(Q,\tau,\sigma)}{\partial \sigma}$ is very low (see Appendix C for details). The right panels of Figure 3 illustrate by converting the implied volatilities into dollar call price quotes in marks. Table A.1 displays some of these values for Dealer I, whose implied volatility quotes appear to differ significantly from the interpolation for the wing options. The largest difference between the dealer's quotes and the interpolation, for very far out-of-the-money dollar calls, is $\frac{3}{100}$ of one pfennig, or about 20 percent. The remaining interpolated values are almost identical to the actual quotes. The effect of the risk-neutral probability density function would be to shift some probability mass to the extreme tail toward which the distribution is skewed. Because there is so little probability mass in the extreme tails, the difference would be imperceptible.

Dealers set implied volatilities for less actively traded deltas by combining an interpolation or parameterization with judgment based on their sense of the market and their wish to posture their option portfolios in particular ways. The differences in wing option volatilities among dealers are related to differences in views about the likelihood of very large moves in exchange rates, in hedging needs, or in anticipated order flows for these options. Since the price differences in currency units are marginal, and dealers do not typically take on large net positions, dealers quoting high do not buy large quantities of options from dealers quoting low. Rather, their wing option quotes express a general posture.

The appearance of the volatility smile in the lower panel of Figure 3 may surprise some readers, since it is concave to the origin for deep in- and out-of-the-money options.[26] This property of volatility smiles is consistent with the no-arbitrage restrictions on option prices.[27]

Work by Heynen (1994) and Taylor and Xu (1994) indicates that volatility smiles in the currency and stock-index option markets can be approximated by quadratic functions of the exercise price or of the ratio of the exercise price to the forward price. However, these studies are based on exchange-traded option data, for which there are relatively few observations on wing options. A

[26]In the example, it is concave for deltas greater than about 0.85 and less than 0.15, about 4 percent or more above and below the current forward rate, as would be typical for flexible exchange rates with modest forward premiums and implied volatilities on the order of 10 percent.

[27]A nontechnical discussion of these restrictions in provided in Appendix C. See Hodges (1996) for a more complete discussion.

more complex functional form which permits the smile slope to taper off to zero as delta approaches zero or one might well be found to fit the data at least as well as the simple quadratic. A dealer interpolating in implied volatility-exercise price space might or might not make the interpolation strictly convex even for high and low deltas, but this would translate into very slight differences in call prices, since even a large change in volatility leads to an insignificant change in price for the wing options.

The interpolation method presented here avoids the problems associated with the alternatives of constructing a spline and regressing the implied volatilities on delta or on exercise prices. As noted, the technique presented here is in fact a cubic spline with parabolic endpoints. However, if the spline is applied to a denser ladder of implied volatilities, the endpoints may behave badly, requiring adjustment by hand. The regression technique, employed by Shimko (1993), has the disadvantage that it may not coincide with the actual data anywhere, and the errors can be substantial.

To summarize, the interpolation method presented here generates highly accurate call prices for most exercise prices. It may be off by small amounts for some very high or low exercise prices. Estimates of the risk-neutral density will thus be highly accurate except for small errors in the extreme tails of the distribution. These errors will be small because the option pricing errors are small, as seen from Table A.1, and occur in the wings, where $\frac{\partial c(t,X,T)}{\partial X}$ is nearly constant and equal to zero or $-e^{-r\tau}$.

As the over-counter currency options markets evolve, trading in further out-of-the money calls and puts is likely to become more active. If dealers were to record prices of 10- or 15-delta risk reversals and strangles, one could calculate the implied volatilities of 10- and 90-delta calls, and use these to develop a version of the interpolation function $\hat{\sigma}(\delta)$ anchored at two additional points in the "wings" of the volatility smile. Campa, Chang and Reider (1997) give an example using weekly data over a one-year observation interval.

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Table 1: Guide to currency units

Abbreviation	Name	Units
£$	Cable (sterling-dollar)	Dollars per pound
$DM	Dollar-mark	Marks per dollar
$¥	Dollar-yen	Yen per dollar
DM¥	Mark-yen	Yen per mark
£DM	Sterling-mark	Marks per pound
£¥	Sterling-yen	Yen per pound

Table 2: Descriptive statistics on exchange rates, option prices and moments

	£$	$DM	$¥	DM¥	£DM	£¥
Log currency changes						
Mean	-0.4	-0.1	-0.3	-0.4	-0.5	-0.7
Minimum	-17.1	-10.6	-11.1	-21.3	-15.9	-19.3
Maximum	8.5	11.2	11.4	18.1	4.5	8.1
Excess returns						
Mean	-0.2	-0.2	-0.2	-0.1	-0.4	-0.4
Minimum	-16.6	-10.5	-10.8	-21.1	-15.9	-18.9
Maximum	8.7	10.7	11.8	18.4	4.3	8.6
Forward premium						
Mean	-0.2	0.2	-0.1	-0.3	-0.0	-0.4
Minimum	-0.7	-0.2	-0.5	-0.5	-0.2	-0.6
Maximum	0.0	0.5	0.1	-0.2	0.2	-0.2
At-the-money volatility						
Mean	10.4	11.8	11.0	10.6	7.6	11.7
Minimum	4.7	7.2	6.7	5.5	2.7	6.0
Maximum	24.0	24.0	18.9	20.0	18.0	20.0
Risk reversal price						
Mean	-0.3	-0.1	-0.5	-0.3	-0.3	-0.3
Minimum	-1.8	-1.4	-2.5	-1.6	-3.5	-1.8
Maximum	0.5	1.0	2.0	1.0	0.5	0.5
Strangle price						
Mean	0.2	0.2	0.3	0.3	0.2	0.3
Minimum	0.0	0.0	-0.2	0.0	-0.1	0.0
Maximum	2.0	0.6	0.7	0.6	2.0	0.9
Standard deviation						
Mean	10.9	12.3	11.5	11.1	8.0	12.2
Minimum	5.3	7.8	7.2	6.1	3.3	6.5
Maximum	25.5	24.7	19.4	20.8	18.6	21.1
Coefficient of skewness						
Mean	0.0	0.0	-0.0	0.0	-0.1	0.0
Minimum	-0.3	-0.2	-0.5	-0.2	-0.7	-0.3
Maximum	0.3	0.4	0.6	0.4	0.3	0.3
Coefficient of kurtosis						
Mean	0.4	0.4	0.6	0.5	0.6	0.5
Minimum	0.0	-0.0	-0.2	-0.0	-0.2	0.0
Maximum	1.6	0.9	1.2	1.4	-3.0	1.4

Notes: Logarithmic changes, excess returns, and forward premium: monthly rates in percent; Option prices: annual volatility terms (percent per annum); Risk-neutral standard deviation: standard deviation of monthly log changes in percent per annum; Risk-neutral coefficients of skewness and kurtosis: monthly terms.

Table 3: Standard tests of the forward bias

$$s_T - s_t = \beta_0 + \beta_1(f_{t,T} - s_t) + \epsilon_T.$$

Currency pair	β_1	$\bar{R}^2$
Cable	4.51	0.05
	1.53	
Dollar-mark	0.85	0.00
	0.60	
Dollar-yen	-3.51	0.04
	-1.75	
Mark-yen	-5.98	0.00
	-0.63	
Sterling-mark	1.30	0.00
	0.63	
Sterling-yen	-4.57	0.01
	-1.18	

Daily, March 31, 1992 to June 11, 1996; t-statistics in italics.

Table 4: Excess returns and implied volatility

Cable $\xi_{£\$,T} = \beta_0 + \beta_1 \, \text{icov}_{£\$,\$DM,t} + \beta_2 (atm_{£\$,t})^2 + \beta_3 \, \text{icov}_{£\$,\$¥,t} + \eta_{£\$,T}$

β_1	β_2	β_3	$\bar{R}^2$
-1.21	-16.17	-15.31	0.13
-0.10	*-2.75*	*-0.91*	

Dollar-mark $\xi_{\$DM,T} = \beta_0 + \beta_1 (atm_{\$DM,t})^2 + \beta_2 \, \text{icov}_{£\$,\$DM,t} + \beta_3 \, \text{icov}_{\$DM,\$¥,t} + \eta_{\$DM,T}$

β_1	β_2	β_3	$\bar{R}^2$
26.81	-6.27	-6.09	0.14
4.20	*-0.41*	*-0.43*	

Dollar-yen $\xi_{\$¥,T} = \beta_0 + \beta_1 \, \text{icov}_{\$DM,\$¥,t} + \beta_2 \, \text{icov}_{£\$,\$¥,t} + \beta_3 (atm_{\$¥,t})^2 + \eta_{\$¥,T}$

β_1	β_2	β_3	$\bar{R}^2$
96.16	-96.73	8.84	0.15
-0.88	*-0.42*	*2.27*	

Daily, March 31, 1992 to June 11, 1996; *t*-statistics in italics.

Table 5: Excess returns and exchange rate moments

$$\xi_T = \beta_0 + \sum_i \beta_{\hat{\sigma},i}\hat{\sigma}_{i,t} + \sum_i \beta_{\hat{\alpha},i}\hat{\alpha}_{i,t} + \sum_i \beta_{\hat{\kappa},i}\hat{\kappa}_{i,t} + \eta_T,$$

$$i = (\$£, \$DM, \$¥, DM¥, £DM, £¥)$$

Currency pair	$\bar{R}^2$
Cable	0.26
Dollar-mark	0.38
Dollar-yen	0.38
Mark-yen	0.40
Sterling-mark	0.28
Sterling-yen	0.29

$$\xi_T = \beta_0 + \sum_i \beta_{\hat{\sigma},i}\hat{\sigma}_{i,t} + \sum_i \beta_{\hat{\alpha},i}\hat{\alpha}_{i,t} + \sum_i \beta_{\hat{\kappa},i}\hat{\kappa}_{i,t} + \beta_f(f_{t,T} - s_t) + \eta_T,$$

$$i = (\$£, \$DM, \$¥, DM¥, £DM, £¥)$$

Currency pair	$\bar{R}^2$
Cable	0.26
Dollar-mark	0.39
Dollar-yen	0.40
Mark-yen	0.41
Sterling-mark	0.31
Sterling-yen	0.30

Daily, March 31, 1992 to June 11, 1996.

Table 6: Excess returns and a subset of exchange rate moments

Cable $\xi_{£\$,T} = \beta_0 + \beta_1\hat{\sigma}_{£\$,t} + \beta_2\hat{\alpha}_{£\$,t} + \beta_3\hat{\kappa}_{£\$,t} + \beta_4\hat{\sigma}_{\$DM,t} + \beta_5\hat{\alpha}_{\$DM,t}$
$+\beta_6\hat{\kappa}_{\$DM,t} + \beta_7\hat{\sigma}_{£DM,t} + \beta_8\hat{\alpha}_{£DM,t} + \beta_9\hat{\kappa}_{£DM,t} + \eta_{£\$,T}$

β_1	β_2	β_3	
-1.85	-0.52	-0.42	
-0.92	*-1.72*	*-2.01*	
β_4	β_5	β_6	
-5.30	-0.70	0.19	
-2.25	*-1.66*	*0.87*	
β_7	β_8	β_9	$\bar{R}^2$
1.71	0.44	0.20	0.19
0.55	*1.11*	*1.23*	

Dollar-mark $\xi_{\$DM,T} = \beta_0 + \beta_1\hat{\sigma}_{\$DM,t} + \beta_2\hat{\alpha}_{\$DM,t} + \beta_3\hat{\kappa}_{\$DM,t} + \beta_4\hat{\sigma}_{\$¥,t} + \beta_5\hat{\alpha}_{\$¥,t}$
$+\beta_6\hat{\kappa}_{\$¥,t} + \beta_7\hat{\sigma}_{DM¥,t} + \beta_8\hat{\alpha}_{DM¥,t} + \beta_9\hat{\kappa}_{DM¥,t} + \eta_{\$DM,T}$

β_1	β_2	β_3	
5.52	-0.46	0.75	
3.00	*-1.05*	*3.14*	
β_4	β_5	β_6	
-1.71	0.06	0.07	
-1.18	*0.20*	*0.30*	
β_7	β_8	β_9	$\bar{R}^2$
4.63	0.35	0.44	0.27
2.19	*0.81*	*4.46*	

Dollar-yen $\xi_{\$¥,T} = \beta_0 + \beta_1\hat{\sigma}_{\$DM,t} + \beta_2\hat{\alpha}_{\$DM,t} + \beta_3\hat{\kappa}_{\$DM,t} + \beta_4\hat{\sigma}_{\$¥,t} + \beta_5\hat{\alpha}_{\$¥,t}$
$+\beta_6\hat{\kappa}_{\$¥,t} + \beta_7\hat{\sigma}_{DM¥,t} + \beta_8\hat{\alpha}_{DM¥,t} + \beta_9\hat{\kappa}_{DM¥,t} + \eta_{\$¥,T}$

β_1	β_2	β_3	
2.06	-0.28	0.68	
0.77	*-0.56*	*2.57*	
β_4	β_5	β_6	
4.68	0.17	0.10	
3.14	*0.50*	*0.34*	
β_7	β_8	β_9	$\bar{R}^2$
0.18	0.46	0.29	0.20
0.06	*1.23*	*1.35*	

Daily, March 31, 1992 to June 11, 1996; *t*-statistics in italics.

Table 6 cont.: Excess returns and a subset of exchange rate moments

Mark-yen $\xi_{DM¥,T} = \beta_0 + \beta_1\hat{\sigma}_{\$DM,t} + \beta_2\hat{\alpha}_{\$DM,t} + \beta_3\hat{\kappa}_{\$DM,t} + \beta_4\hat{\sigma}_{\$¥,t} + \beta_5\hat{\alpha}_{\$¥,t} + \beta_6\hat{\kappa}_{\$¥,t} + \beta_7\hat{\sigma}_{DM¥,t} + \beta_8\hat{\alpha}_{DM¥,t} + \beta_9\hat{\kappa}_{DM¥,t} + \eta_{DM¥,T}$

β_1	β_2	β_3	
7.65	-0.67	1.34	
2.07	*-0.78*	*2.99*	
β_4	β_5	β_6	
1.39	0.25	0.21	
0.59	*0.42*	*0.43*	
β_7	β_8	β_9	$\bar{R}^2$
6.14	0.76	0.63	0.24
1.50	*1.16*	*1.69*	

Sterling-mark $\xi_{£DM,T} = \beta_0 + \beta_1\hat{\sigma}_{£\$,t} + \beta_2\hat{\alpha}_{£\$,t} + \beta_3\hat{\kappa}_{£\$,t} + \beta_4\hat{\sigma}_{\$DM,t} + \beta_5\hat{\alpha}_{\$DM,t} + \beta_6\hat{\kappa}_{\$DM,t} + \beta_7\hat{\sigma}_{£DM,t} + \beta_8\hat{\alpha}_{£DM,t} + \beta_9\hat{\kappa}_{£DM,t} + \eta_{£DM,T}$

β_1	β_2	β_3	
-0.83	0.02	-0.40	
-0.52	*0.06*	*-2.51*	
β_4	β_5	β_6	
-0.44	-0.63	1.14	
-0.22	*-2.84*	*5.16*	
β_7	β_8	β_9	$\bar{R}^2$
0.38	0.33	-0.26	0.19
0.17	*0.97*	*-1.92*	

Sterling-yen $\xi_{£¥,T} = \beta_0 + \beta_1\hat{\sigma}_{£\$,t} + \beta_2\hat{\alpha}_{£\$,t} + \beta_3\hat{\kappa}_{£\$,t} + \beta_4\hat{\sigma}_{\$¥,t} + \beta_5\hat{\alpha}_{\$¥,t} + \beta_6\hat{\kappa}_{\$¥,t} + \beta_7\hat{\sigma}_{£¥,t} + \beta_8\hat{\alpha}_{£¥,t} + \beta_9\hat{\kappa}_{£¥,t} + \eta_{£¥,T}$

β_1	β_2	β_3	
2.06	-0.28	0.68	
0.77	*-0.56*	*2.57*	
β_4	β_5	β_6	
4.68	0.17	0.10	
3.14	*0.50*	*0.34*	
β_7	β_8	β_9	$\bar{R}^2$
0.18	0.46	0.29	0.23
0.06	*1.23*	*1.35*	

Daily, March 31, 1992 to June 11, 1996; t-statistics in italics.

Table 7: The forward premium and skewness

$$f_{t,T} - s_t = \beta_0 + \beta_1 \alpha_t + \epsilon_T.$$

Currency pair	$\beta_1 * 10^3$	$\bar{R}^2$
Cable	2.56	0.02
	1.49	
Dollar-mark	6.37	0.09
	2.63	
Dollar-yen	-1.53	0.02
	-1.41	
Mark-yen	0.59	0.01
	1.10	
Sterling-mark	3.73	0.12
	3.90	
Sterling-yen	-2.79	0.01
	-2.85	

Daily, March 31, 1992 to June 11, 1996; *t*-statistics in italics.

Table 8: Skewness and the forward bias

$$s_T - s_t = \beta_0 + \beta_1(f_{t,T} - s_t) + \beta_2\alpha_t + \epsilon_T.$$

Currency pair	β_1	β_2	$\bar{R}^2$
Cable	5.27	-0.04	0.07
	1.40	*-1.22*	
Dollar-mark	-0.29	0.06	0.04
	-0.22	*2.16*	
Dollar-yen	-4.58	0.03	0.08
	-2.30	*1.25*	
Mark-yen	-5.70	0.02	0.00
	-0.60	*0.35*	
Sterling-mark	0.88	0.02	0.01
	0.39	*0.59*	
Sterling-yen	-2.23	0.08	0.03
	-0.54	*1.68*	

Daily, March 31, 1992 to June 11, 1996; *t*-statistics in italics.

Table 9: Skewness and excess returns

$$\xi_T = \beta_0 + \beta_1 \alpha_t + \epsilon_T.$$

Currency pair	β_1	$\bar{R}^2$
Cable	-0.36	0.01
	-0.84	
Dollar-mark	0.61	0.03
	1.79	
Dollar-yen	0.41	0.03
	1.41	
Mark-yen	0.22	0.00
	0.29	
Sterling-mark	0.24	0.01
	0.61	
Sterling-yen	1.03	0.03
	1.96	

Daily, March 31, 1992 to June 11, 1996; t-statistics in italics.

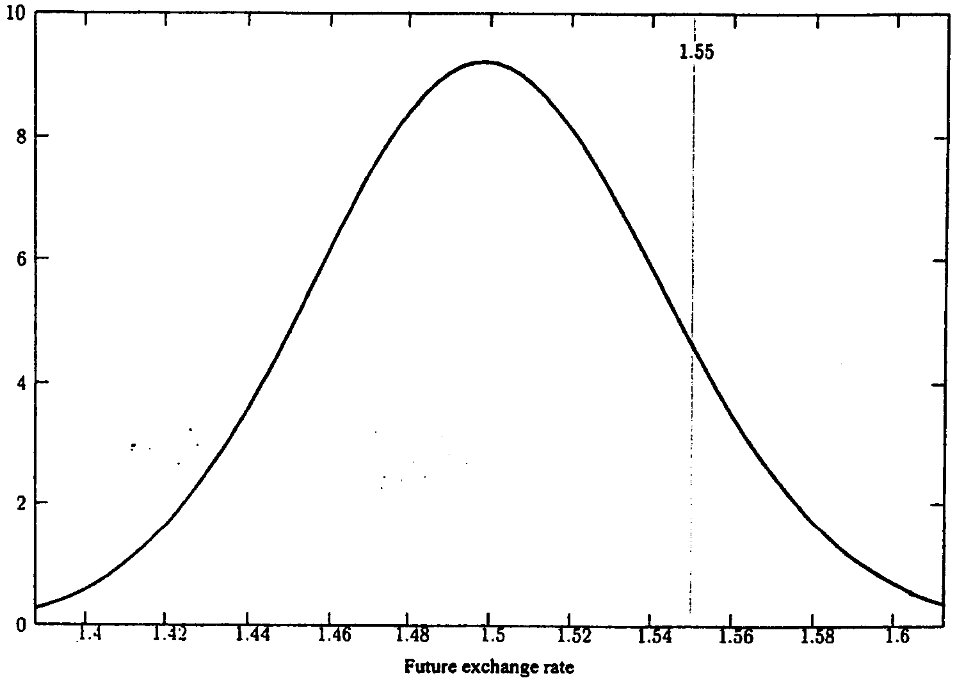
Table A.1: Dealer quotes and interpolated option values

Based on sample dealer's indicative levels for one-month dollar-mark calls, Mar. 3, 1997

Call delta	Dealer quote	Interpolation
0.05	0.0011	0.0014
0.10	0.0025	0.0028
0.25	0.0078	0.0078
0.40	0.0145	0.0144
0.60	0.0270	0.0270
0.75	0.0416	0.0416
0.90	0.0698	0.0697
0.95	0.0878	0.0878

Figure 1

Call option values and the risk-neutral probability density

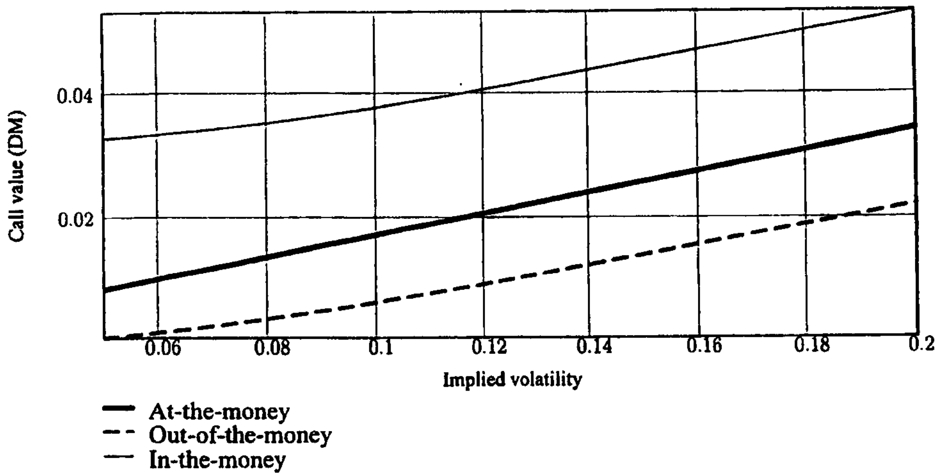


The area under the risk-neutral density to the right of a vertical line at DM 1.55 equals the market price of a European call on the dollar with an exercise price of DM 1.55 and a maturity of τ years.

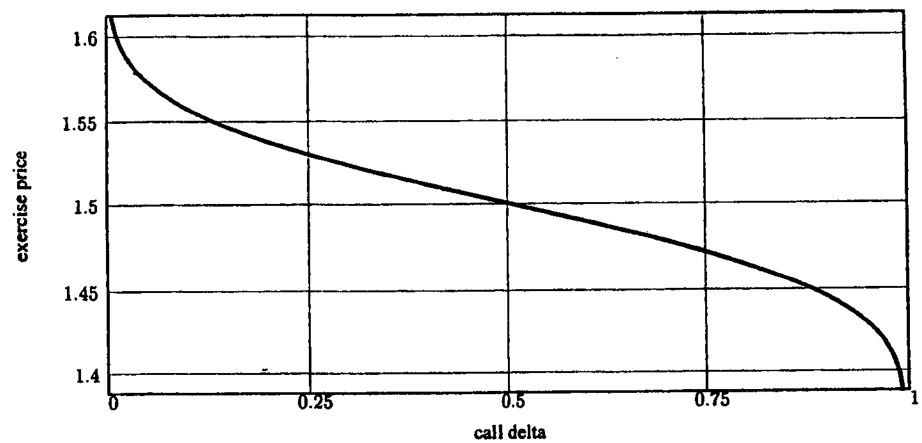
Figure 2

Over-the-counter currency option market conventions

a. Call value and implied volatility



b. Call option delta and exercise price

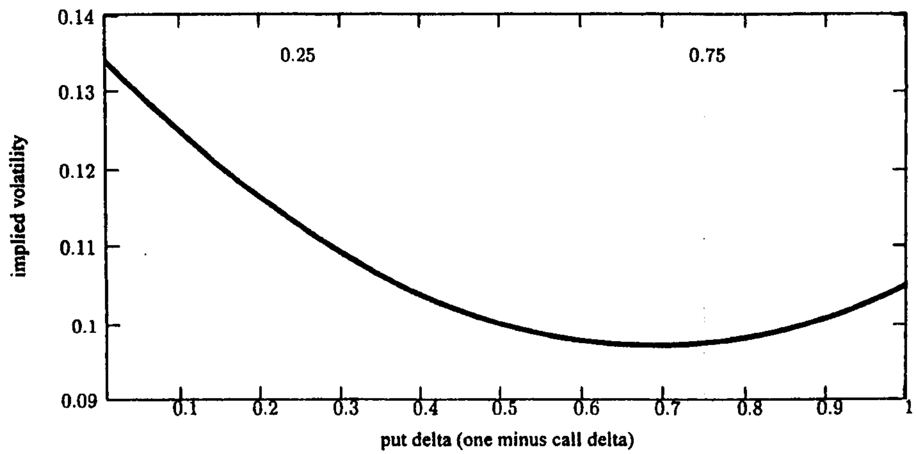


Dollar call against the German mark, spot exchange rate DM 1.50, maturity 1 month, implied volatility 10 percent. Interest rates: domestic 5 percent, foreign 5 percent.

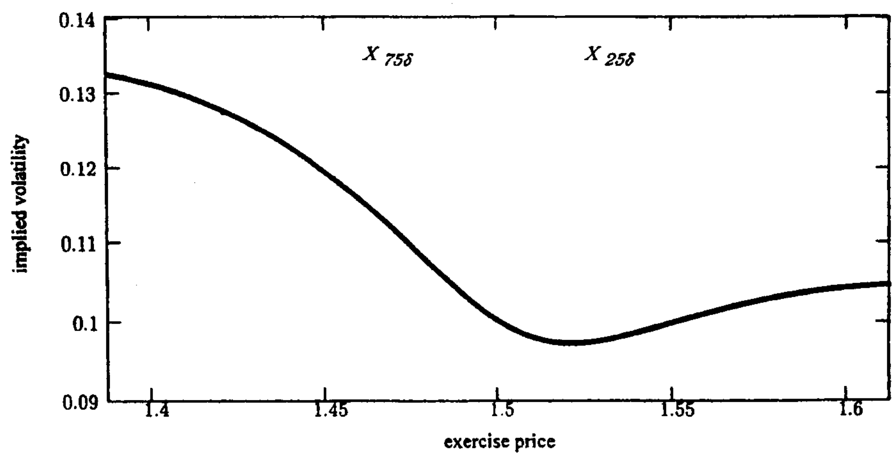
Figure 3

The volatility smile

a. Volatility smile as a function of delta



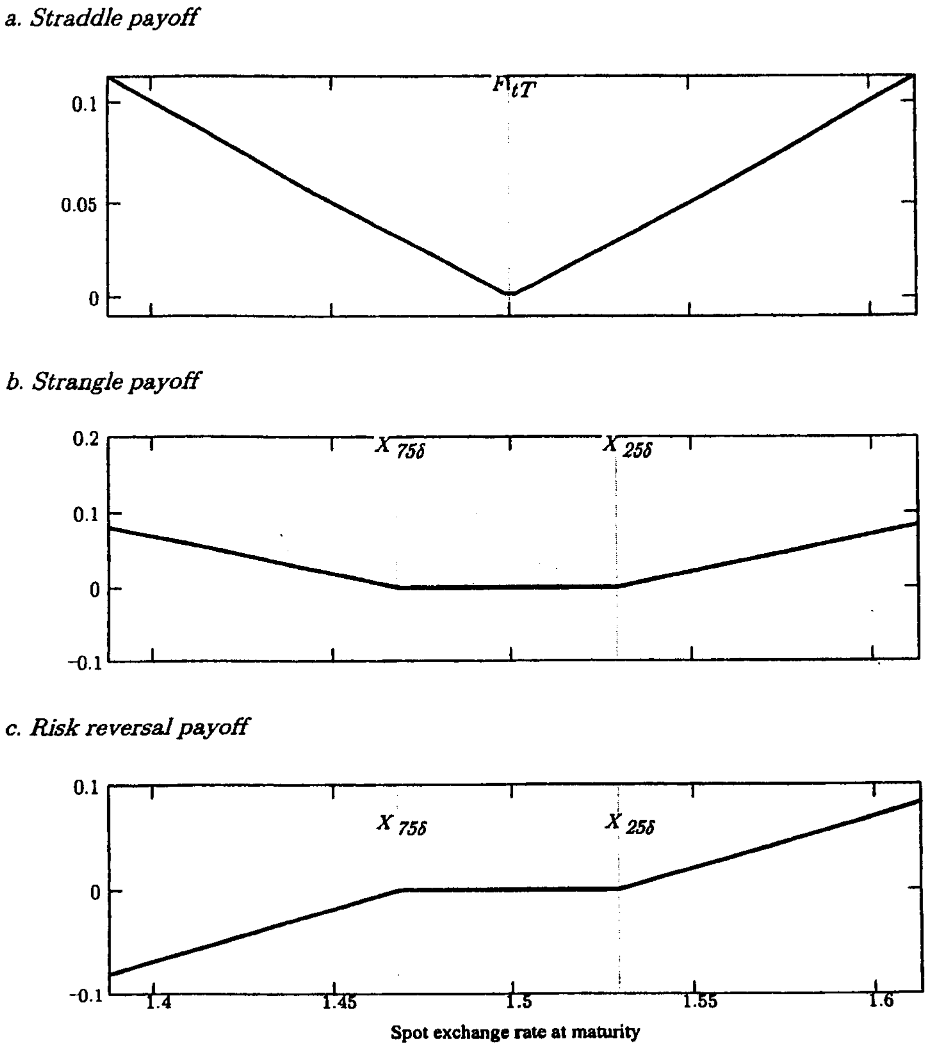
b. Volatility smile as a function of exercise price



Dollar call against the German mark, spot exchange rate DM 1.50, maturity 1 month, implied volatility 10 percent. Interest rates: domestic 5 percent, foreign 5 percent.

Figure 4

Straddle, strangle and risk reversal payoffs at maturity

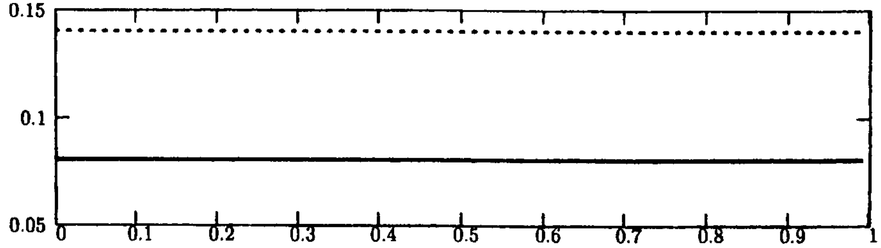


Spot exchange rate DM 1.50; payoff in DM. F_{tT}:= one-month forward exchange rate, $X_{25\delta}$:= exercise price of 25-delta call, $X_{75\delta}$:= exercise price of a 75-delta call.

Figure 5

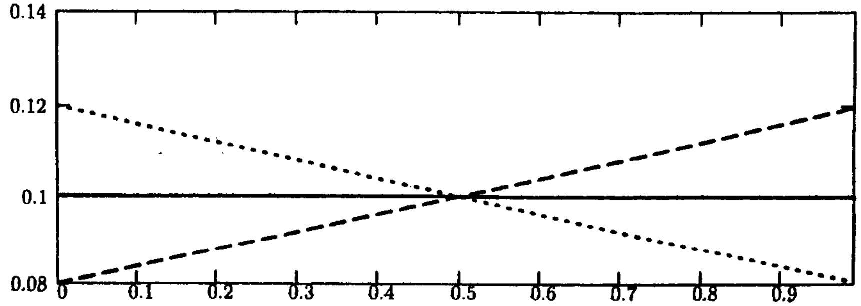
Components of the volatility smile

a. Constant equal to at-the-money volatility



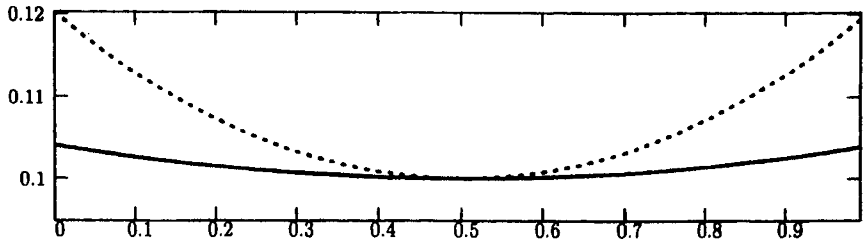
Solid line: atm_t=8%; dotted line atm_t=14%.

b. Linear in risk reversal price



Solid line: rr_t=0%; dotted line rr_t=+2%; dashed line rr_t=-2%.

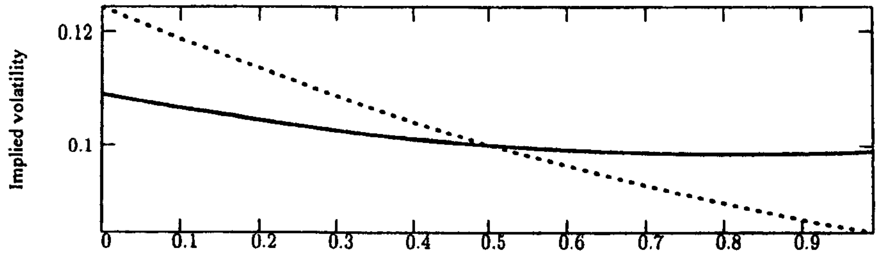
c. Quadratic in strangle price



Solid line: str_t=0.1%; dotted line str_t=0.5%.

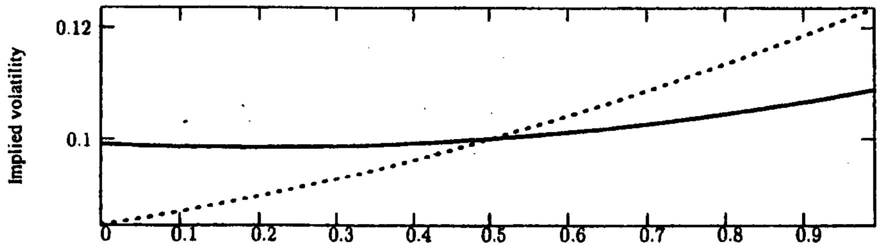
Figure 6
Common types of volatility smiles

a. Positive skew, minor kurtosis



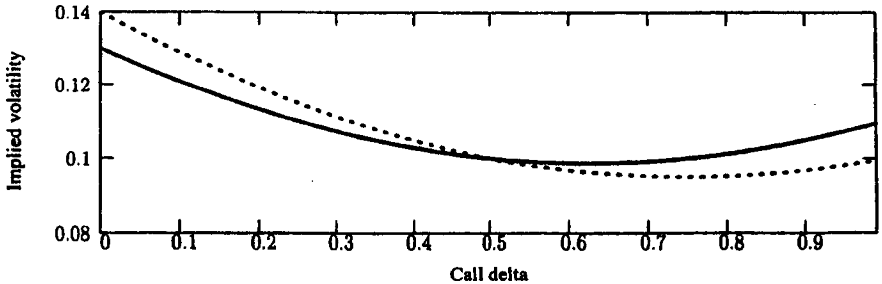
Solid line: rr_t=0.5%, str_t=0.1%; dotted line rr_t=2.0%, str_t=0.1%.

b. Negative skew, minor kurtosis



Solid line: rr_t=-0.5%, str_t=0.1%; dotted line rr_t=-2.0%, str_t=0.1%.

c. Highly skewed and kurtotic



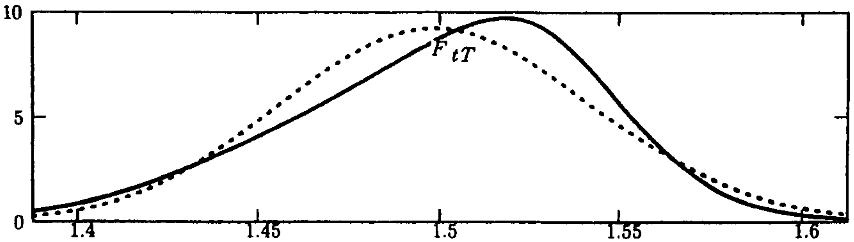
Solid line: rr_t=1.0%, str_t=0.5%; dotted line rr_t=2.0%, str_t=0.5%.

All examples assume atm_t=10.0%.

Figure 7

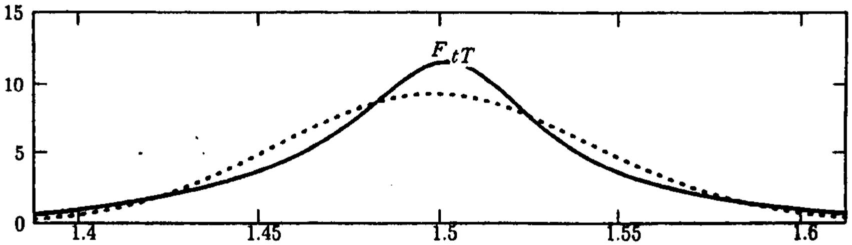
Estimated risk-neutral distributions

a. Skewed but not kurtotic



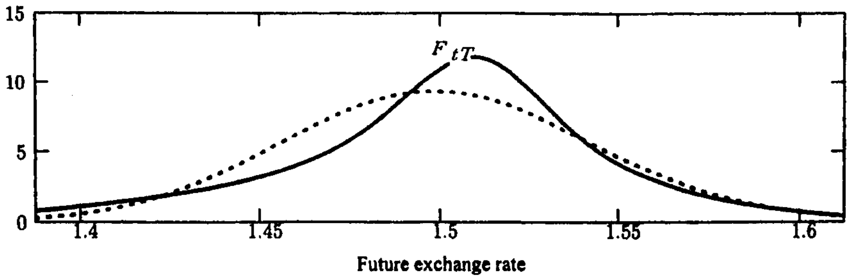
Assumes rr_t=-1.5%, str_t=0.0%.

b. Kurtotic but not skewed



Assumes rr_t=0%, str_t=0.5%.

c. Skewed and kurtotic



Assumes rr_t=-1.5%; str_t=0.5%.

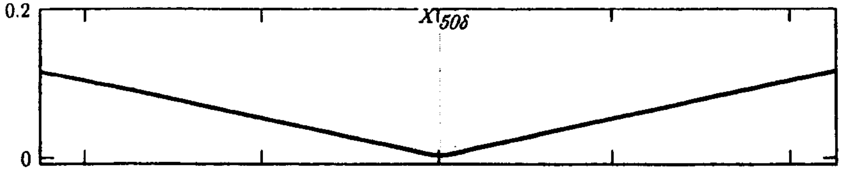
Dollar call against the German mark, spot exchange rate DM 1.50, maturity 1 month, implied volatility 10 percent. Interest rates: domestic 5 percent, foreign 5 percent.

All examples assume atm_t=10.0% Compare with lognormal (dotted line) with σ=10.0%.

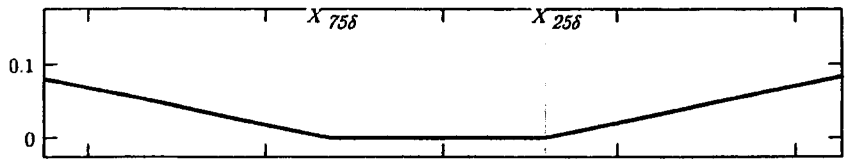
Figure 8

Option payoffs and the risk-neutral probability density

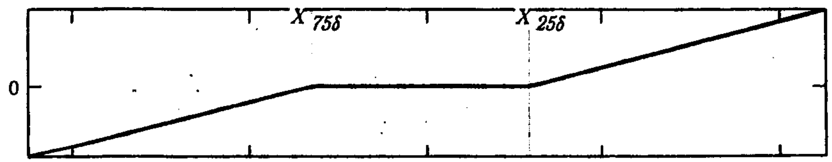
a. Strangle payoff



b. Strangle payoff



c. Risk reversal payoff



d. Risk-neutral probability density

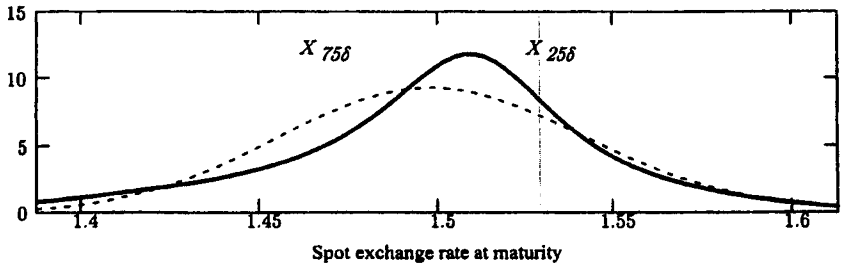
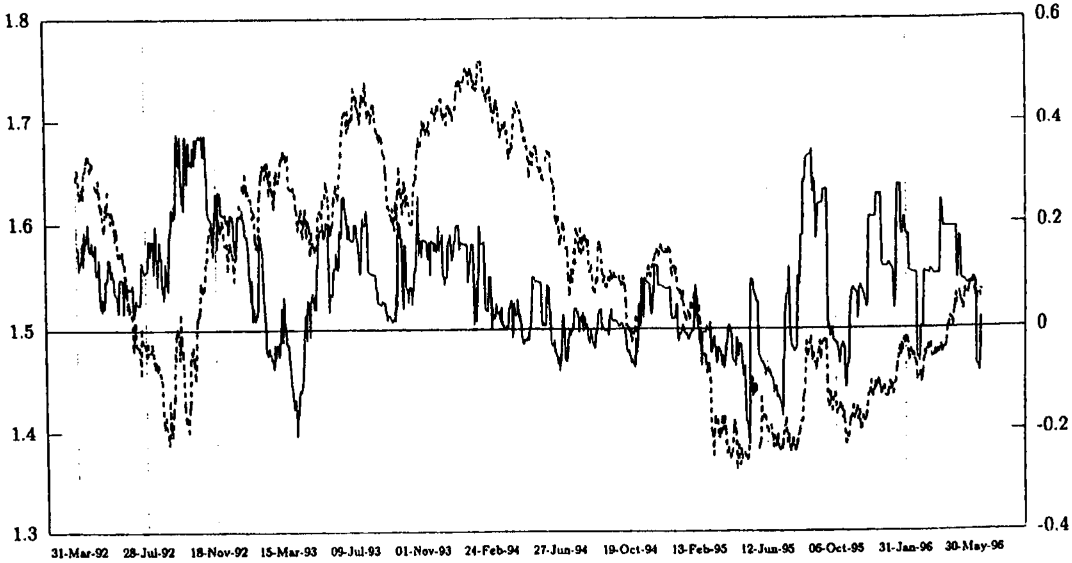


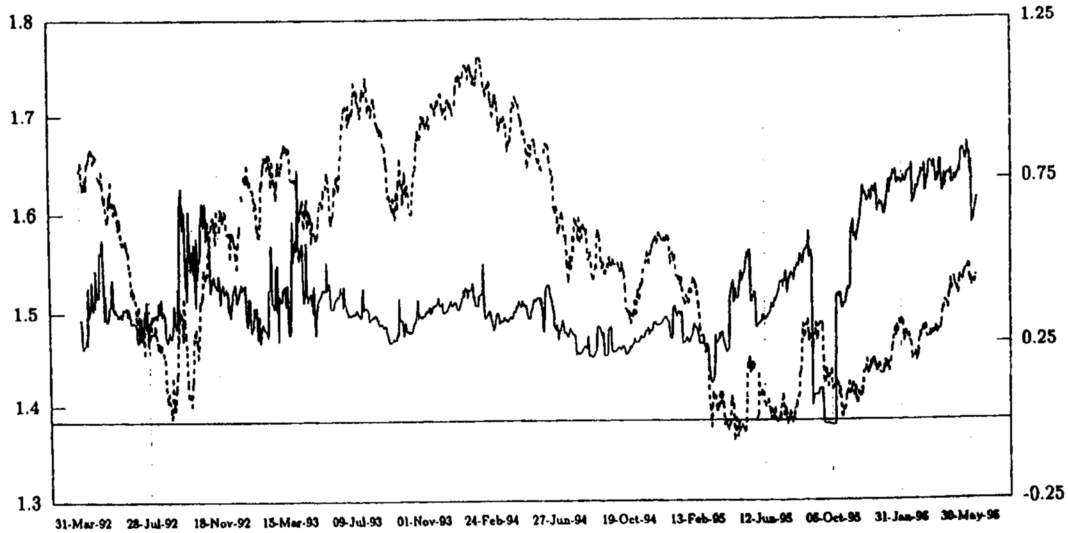
Figure 9

Skewness coefficient of the dollar-mark exchange rate



-- Spot exchange rate (left axis) — Coefficient of skewness (right axis)

Kurtosis coefficient of the dollar-mark exchange rate

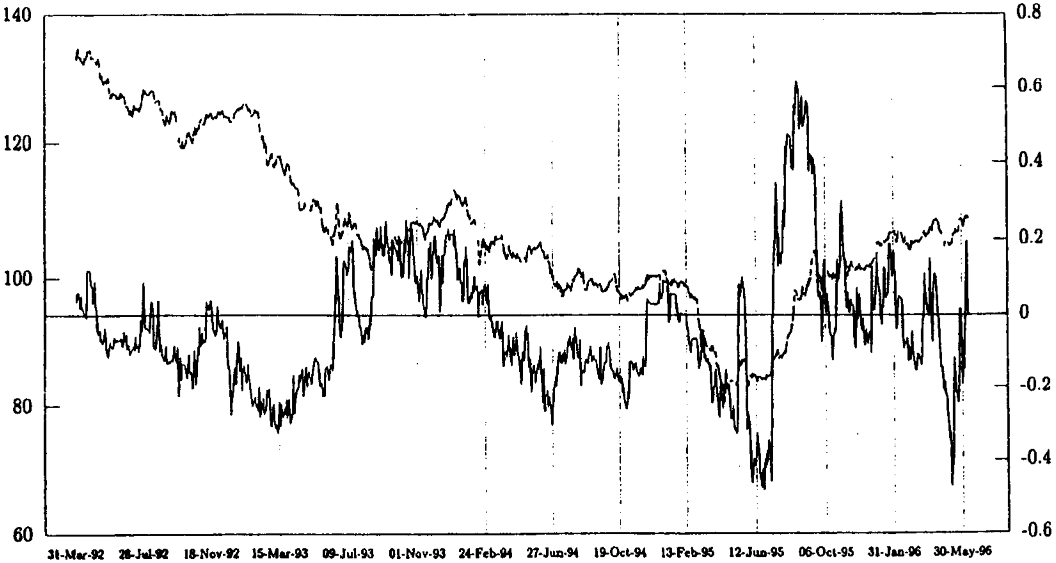


-- Spot exchange rate (left axis) — Coefficient of kurtosis (right axis)

Daily, March 31, 1992 to June 11, 1996

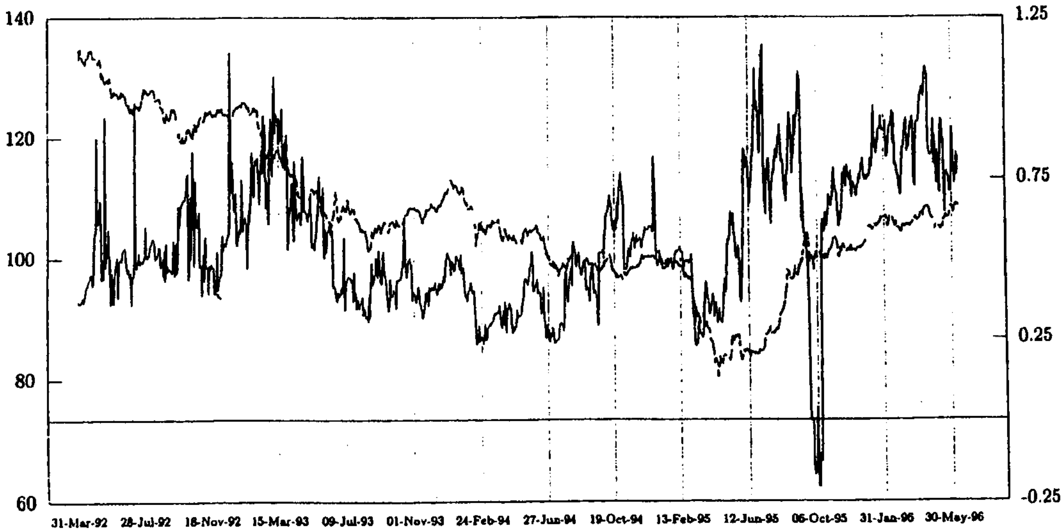
Figure 10

Skewness coefficient of the dollar-yen exchange rate



-- Spot exchange rate (left axis) — Coefficient of skewness (right axis)

Kurtosis coefficient of the dollar-yen exchange rate

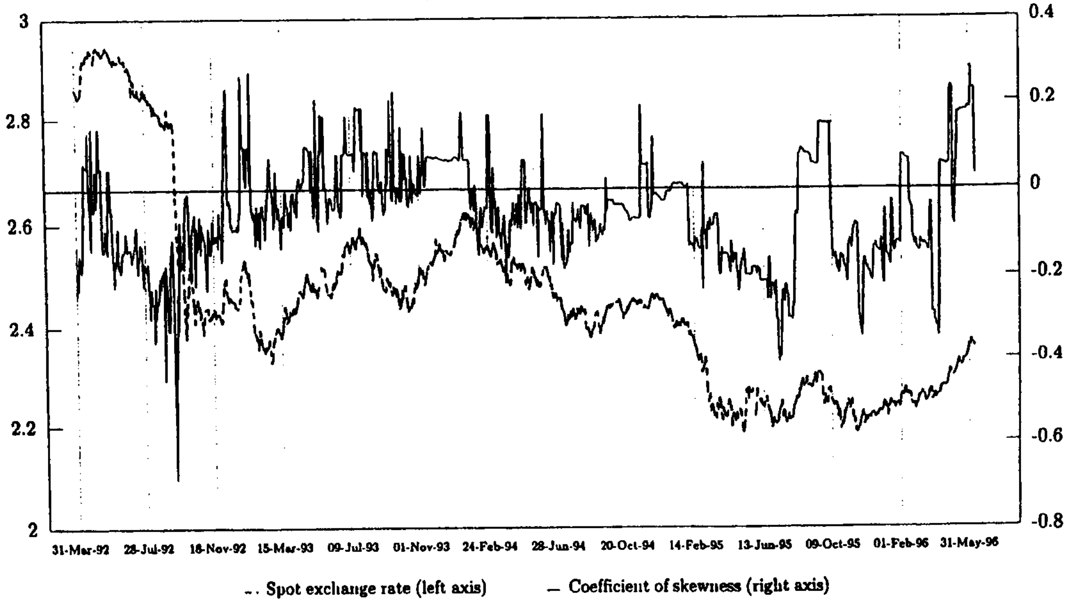


-- Spot exchange rate (left axis) — Coefficient of kurtosis (right axis)

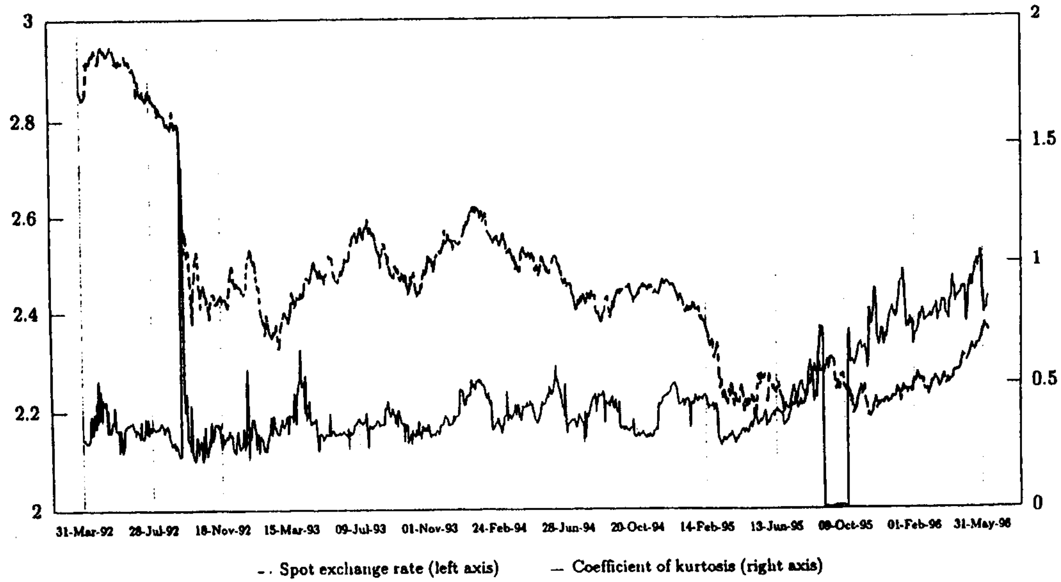
Daily, March 31, 1992 to June 11, 1996

Figure 11

Skewness coefficient of the sterling-mark exchange rate



Kurtosis coefficient of the sterling-mark exchange rate

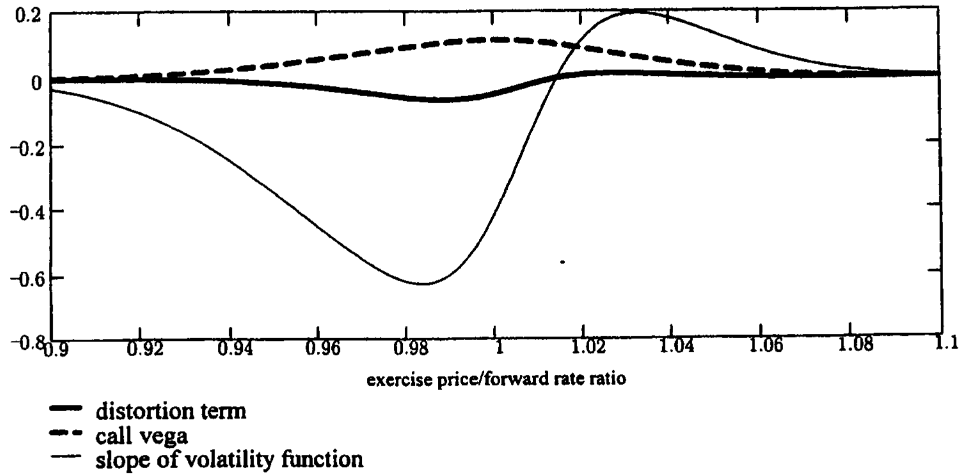


Daily, March 31, 1992 to June 11, 1996

Figure A.1

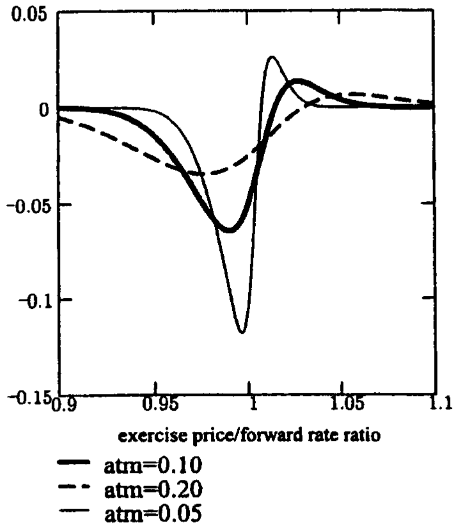
Slope of the volatility function

a. Components of the smile distortion term



Assumes atm_t=10%, rr_t=-1.5%, str_t=0.5%.

b. Smile distortion term for varying atm



c. Smile distortion term for varying rr

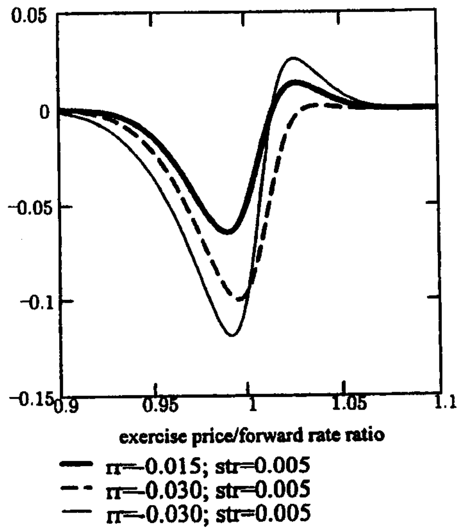
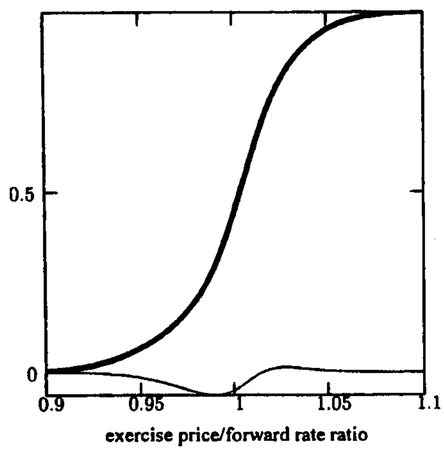


Figure A.2

Smile distortion term and cumulative distribution function

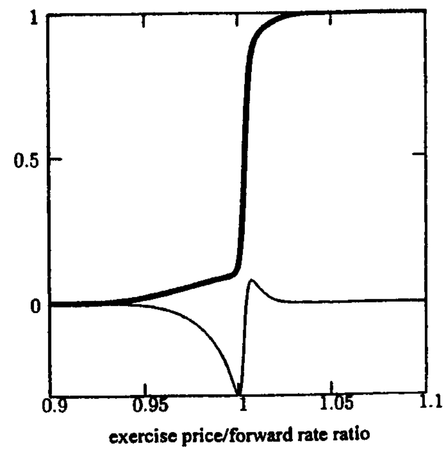
a. Typical flexible exchange rate

atm = 0.10; *rr* = -0.015; *str* = 0.005



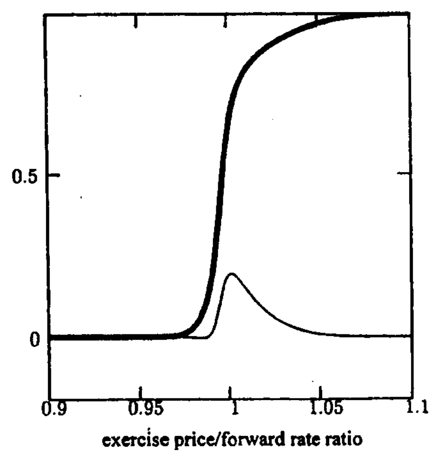
b. "ERM exchange rate"

atm = 0.005; *rr* = -0.03; *str* = 0.01



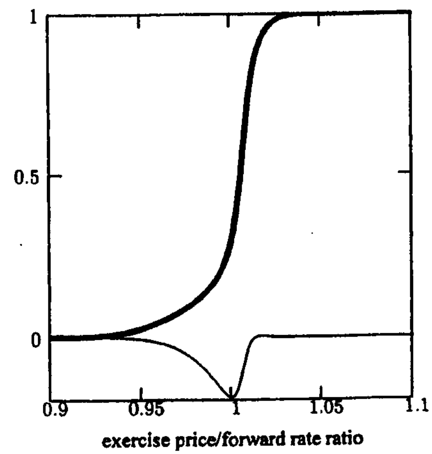
c. Negative skew

atm = 0.10; *rr* = -0.03; *str* = 0.01



d. Positive skew

atm = 0.10; *rr* = 0.03; *str* = 0.01

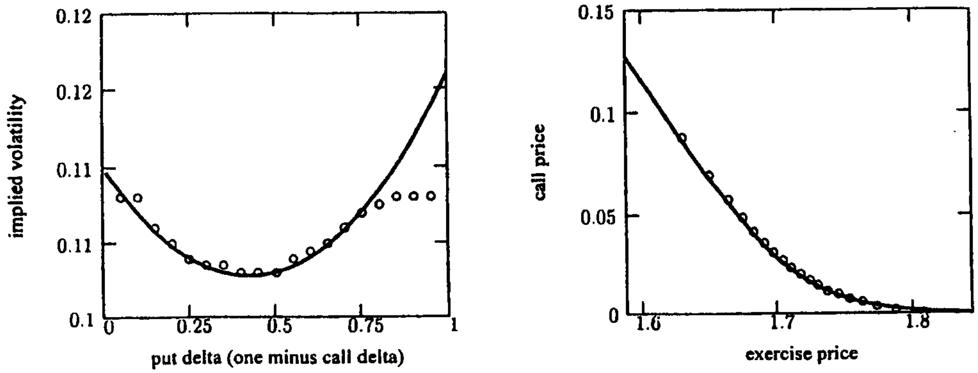


Heavy lines represent cumulative distribution functions derived by the method described in the tex
Thin lines represent the smile distortion term.

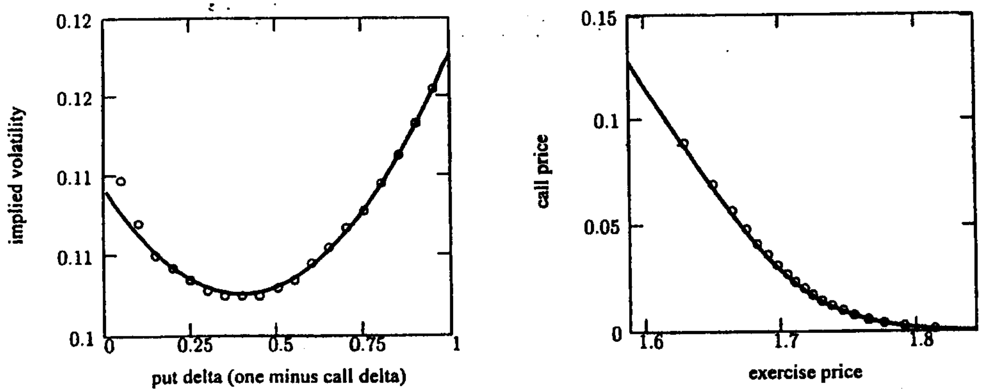
Figure A.3

Dealer quotes and interpolated option values

a. Dollar-mark, dealer I, March 6, 1997



b. Dollar-mark, dealer II, March 6, 1997



Circles represent dealer quotes and solid lines represent values derived by interpolation.

Dollar call against the German mark, spot exchange rate DM 1.7160, maturity 1 month. Interest rates: domestic (USD) 3.33 percent, foreign (DEM) 5.48 percent.